Math 2150 Notes - June 13, 2022

One of the central concepts in mathematics is that of an *assertion*. One of the main aims of theoretical mathematics is to evaluate the truth/falsity of various assertions, also known as propositions. We give the standard definition below.

Definition 1. *A **proposition** is a statement which is unambiguously true or false (but not both).*

We present an example below.

Example 1. *Determine if the following are propositions. For those which are propositions, state whether they are true or false.*

1. *Denver is the capital of Colorado.*
2. $2 + 2 = 6$.
3. *Turn off the water!*
4. $x + y = 4$.
5. *This sentence is false.*
6. *Pizza tastes good.*

Solution We evaluate each in succession.

(1) "Denver is the capital of Colorado" is an unambiguously true statement, and hence is a proposition.

(2) "$2+2 = 6$" is an unambiguously false statement, and hence is a proposition (REMEMBER: a proposition does not need to be true; it simply must be true OR false).

(3) "Turn off the water!" is a command; it is not making an assertion, but rather a *request*, so this is not a proposition.

(4) "$x + y = 4$" is NOT a proposition. The issue here is that we are not given what x and y are. If x and y both happen to be 2, then this is a true proposition, but if $x = 6$ and $y = 3$, then the assertion is false. The issue here is that the assertion is not *unambiguously* true or false; its truth value depends on the value of x and y, and so this is NOT a proposition.

(5) "This sentence is false" is not a proposition. This is a somewhat famous example of a so-called paradox. If the sentence is true, then the sentence is false, based on what the sentence is asserting. But if "This sentence is false" is false, then the sentence is true. We generally won't be going into paradoxes in this course. If you are interested in reading more about these, you may want to look up the so-called "barber paradox".

(6) "Pizza tastes good" is an opinion, and it isn't shared by everyone (hard to imagine, but true), so this is NOT a proposition. □

In what follows (unless specified otherwise), we will use lowercase letters (p,q,r,s, etc.) to denote propositions. These letters will be called **propositional variables**. We will now spend a bit of time constructing new propositions from existing ones by introducing so-called **logical operators**. As a quick reminder, a proposition p is an assertion which is unambiguously true or false, but not both. I will be introducing some very basic so-called **truth tables**, but we won't be studying them in detail until a bit later. For now, don't worry if you don't understand what they are saying; we will go into specifics shortly.

Definition 2. *Let p be a proposition. Then the proposition **not** p, denoted $(\neg p)$ and read "not p", is the assertion "p is false".*

Example 2. *Let p be the proposition "Joe Biden is the president of the United States". Then $(\neg p)$ translates to "Joe Biden is not the president of the United States". Note that p is true, but $(\neg p)$ is false.*

Example 3. *Let p be the proposition "two plus two equals five". Then $(\neg p)$ translates to "two plus two does not equal five". Note that p is false, but $(\neg p)$ is true.*

REMARK: We may extrapolate the following more general principle from the previous two examples: If p is a true proposition, then $(\neg p)$ is false; if p is a false proposition, then $(\neg p)$ is true. We me express this more succinctly using the following true table.
Truth table for the "not" operator:

p	$(\neg p)$
T	F
F	T

We now come to our next logical operator. We call this a **binary** operator because the input consists of two propositional variables instead of just one.

Definition 3. *Let p and q be propositions. Then the proposition p **and** q, denoted $(p \wedge q)$ and read "p and q", is the assertion that "both p and q are true."*

Example 4. *Let p be "$2 \times 3 = 6$" and q be "Canada is a continent". Note that p is true but q is false. Thus $(p \wedge q)$ is false precisely because it is NOT the case that p and q are BOTH true.*

I trust this concept is sufficiently clear that we don't need more examples. Here is a truth table for the "and" operator. Note that the first two columns simply give all possible true values for the propositional variables p and q, and the final column gives the truth value of $(p \wedge q)$ based on the truth values of p and q which precede it.

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 4. *Let p and q be propositions. Then the proposition p **or** q, denoted $(p \vee q)$ and read "p or q", is the assertion that "either p is true or q is true OR BOTH ARE TRUE."*

Example 5. *Let p be "all dogs are animals" and q be "the earth is the first planet from the sun". Then p is true but q is false, so at least one of p, q is true. Hence $(p \vee q)$ is true.*

Example 6. *Let p be "$2 - 4 = -2$" and q be "$3 \cdot 4 = 12$". Then both p and q are true, so at least one of p, q is true. Hence $(p \vee q)$ is true in this case as well.*

The upshot: $(p \vee q)$ is ONLY FALSE when *BOTH* p and q are false. Here's the truth table:

p	q	$(p \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

Our next operator defines *implication* which is ubiquitous in mathematics. Before introducing the operator formally, I'd like to give an example from real life. I really would like you to take a few minutes to try to *feel* why the following is true.

Example 7. *Suppose I tell you the following: "If it's not raining tomorrow night, I'll meet you for dinner at Rioja at 7 pm." Now, suppose 7 pm comes around tomorrow, it isn't raining, and yet I am nowhere to be found at Rioja. Then you may think that I lied to you (or at minimum, I didn't fulfill my commitment). Why? Let's dissect this conditional statement. Let p be "it's not raining tomorrow night" and q be "I'll meet you for dinner at Rioja at 7 pm (tomorrow)." Then observe that in the above scenario, p was TRUE (it, in fact, didn't rain tomorrow night) but q was FALSE (I didn't show up). This example will motivate the following definition.*

Definition 5. *Let p and q be propositions. Then the conditional proposition* ***if p, then*** *q, denoted $(p \to q)$ and read "if p, then q", is ONLY FALSE IF p is true and q is false. I CANNOT OVERSTATE HOW IMPORTANT IT IS FOR YOU TO COMMIT THIS TO MEMORY* ***NOW!!!****. The motivation here is that "if p then q" is saying that the truth of p, in some sense,* forces *the truth of q. If p happens to be false, then no forcing can happen, and in this case, we simply declare "if p, then q" to be true by fiat.*

Example 8. *Let p be "$2 - 5 = -3$" and q be "$0 \cdot 10 = 5$". Then note that p is true but q is false, and hence $(p \to q)$ is false.*

Here is the truth table:

p	q	$(p \to q)$
T	T	T
T	F	F
F	T	T
F	F	T

Our final logical operator is the "if and only if" operator, often abbreviated in mathematics as "iff". Again, I urge you all to commit the following definition to memory.

Definition 6. *Let p and q be propositions. Then the conditional proposition p* ***if and only if*** *q, denoted $(p \leftrightarrow q)$ and read "p if and only if q", is the assertion that either p and q are both true OR p and q are both false. Said more succinctly, $(p \leftrightarrow q)$ is true exactly when p and q have the same truth value.*

Example 9. *Let p be "a cat is an insect" and q be "$2 + 3 = 5$". Then note that p is false but q is true. So p and q do NOT have the same truth value, and thus $(p \leftrightarrow q)$ is false.*

Example 10. *Let p be "$11 = 4$" and q be "$2 + 3 = 89$". Then both p and q are false, and hence p and q have the same truth value, so $(p \leftrightarrow q)$ is true.*

Here is the truth table:

p	q	$(p \leftrightarrow q)$
T	T	T
T	F	F
F	T	F
F	F	T

We have now seen the five fundamental logical operators: $\neg$ (not), $\wedge$ (and), $\vee$ (inclusive or), $\to$ (if, then), and $\leftrightarrow$ (if and only if). Some of you may have seen other logical operators in a computer science or philosophy course. Indeed, there are many more. However, all other logical operators can be expressed using only the logical operators above. In fact, even the above list is redundant. We can, in fact, express every Boolean operator using only the $\neg$ and $\vee$ operators. If you don't know what I'm talking about, don't worry; I'm just trying to give some extra information for those of you who do.

Today, we will introduce the formal syntax of propositional formulas and also study these formulas semantically. For reference, much of (though not all) what I will be sharing can be augmented in section 1.1 of the Rosen text I sent yesterday.

0.1 Syntax

[1] Since this material is not presented in the text, I will present several examples to help you intuit this information. For now, I want you to focus **only** on the syntax of the formulas we will introduce; we will discuss how to interpret the formulas in the next subsection.

Definition 7 (Inductive definition of formulas)**.** *The following rules govern the construction of propositional* ***formulas****.*

1. *Every propositional variable is a formula.*
2. *If α and β are formulas, then so are $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$.*
3. *No string of symbols is a formula unless compelled to be so by repeated application of (1) and (2).*

In the next examples, we justify why certain expressions are formulas. You will also have a bit of this to do in the homework, just fyi.

Example 11. *Let p be a propositional variable. Then p is a formula.*

The reason is simple: this holds by 1. of the above definition.

Example 12. *Let p and q be propositional variables. Then $(p \vee q)$ is a formula.*

Now things get just a bit more complicated: first, by 1., both p and q are formulas, since they are propositional variables. Applying 2., we see that $(p \vee q)$ is also a formula (take α to be p and β to be q).

Example 13. *Let p be a propositional variable. Then $(\neg(\neg p))$ is a formula.*

Since p is a propositional variable, p is a formula. Now applying 2., we see that $(\neg p)$ is a formula. By 2. again (with α being $(\neg p)$), we see that $(\neg(\neg p))$ is a formula.

Example 14. *Let p, q, and r be propositional variables. Then $((p \vee q) \to r)$ is a formula.*

To see why this is true, note first that p and q are formulas by 1. Now by 2., $(p \vee q)$ is a formula. By 1., r is a formula. Finally, by 2. (with α being $(p \vee q)$ and β being r), it follows that $((p \vee q) \to r)$ is a formula.

Remark 1. *The general idea is as follows: begin with the variables, and then "build up" the formula from the inside out, making sure to justify each step along the way by appealing to definitions. We will see more examples shortly.*

Remark 2. *(IMPORTANT) The text indicates certain abbreviations which allows one to not include so many parentheses. We will get to this later, but for now I do NOT want you to do this. I want you to follow the algorithm precisely. The reason is simple: many of you will be doing some coding either in computer science, math, or engineering, and if you can't get syntax right, you're likely to get garbage as output.*

0.2 Semantics and More on Truth Tables

First, consider propositional variables p and q. *Without context*, it doesn't make sense to say that, say, $(p \vee q)$ is *true*; similarly, it makes no sense to say that $(p \vee q)$ is *false*. For example, suppose that p is "$2 + 2 = 5$" and q is "Thomas Jefferson was the first president of the United States". Then $(p \vee q)$ is false. Similarly, if p is "$2 + 2 = 4$" and q is "The speed limit on I-25 is 200 mph", then $(p \vee q)$ is true. The moral: the truth value of $(p \vee q)$ *is, in general, a function of the truth values of p and q.*[2]

Recall from the August 24 notes that I introduced truth tables for simple formulas involving the five logical operators (you may want to scroll back up and take a look at these now). The tables gave the truth value of the simple formulas as a function of the truth values of the variables from which the formulas were built. We can extend this algorithm to compound formulas as well. We will start our study in a very algorithmic way, but I hope to impart some intuition as well before concluding the lecture.

[1] This subsection is NOT presented in the textbook.

[2] There are exceptions; these will be discussed shortly.

Algorithm 1 (Instructions for Constructing Truth Tables)**.** *To construct a truth table for a propositional formula α,*

1. *List the variables which appear in the formula (from left to right).*
2. *List the **subformulas** of α (not equal to variables) from left to right, with the given formula at the very right.*
3. *Fill in all combinations of the truth values of the variables.*
4. *Use this values to complete the table, proceeding down columns from left to right.*

Note the word "subformula" above. For this course, you may take "subformula" to mean simply a formula you come to as you justify (using the inductive definition) that a given formula really is a formula. Before diving into truth tables, let's look at another example.

Example 15. *Let p and q be propositional variables. Then $((\neg p) \vee q)$ is a formula.*

To see why, observe that p and q are variables, hence formulas by 1. of the definition of "formula". By 2. of the definition, $(\neg p)$ is a formula. Now we have that both $(\neg p)$ and q are formulas, so by 2. again, $((\neg p) \vee q)$ is a formula. We now use our work above to construct a truth table for $((\neg p) \vee q)$:

Example 16.

p	q	$(\neg p)$	$((\neg p) \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

A few words about the final column: the formula that heads this column is $((\neg p) \vee q)$. Look at the first row of truth values. We have "F" for $(\neg p)$ and "T" for q. Now, $((\neg p) \vee q)$ is true precisely when AT LEAST ONE of $(\neg p)$ and q is true, and we have this since q is true. Thus the value of the formula is also "T" in this case, and we record it below. Similar remarks apply to the final three rows of the table.

Now let's take a look at a formula with three variables.

Example 17. *Let p, q, and r be propositional variables. Then $((p \vee q) \wedge r)$ is a formula.*

For justification, note that p, q, and r are formulas by 1. of the definition. Now by 2. of the definition, $(p \vee q)$ is a formula. Since $(p \vee q)$ and r are formulas, we may apply 2. again to conclude that $((p \vee q) \wedge r)$ is a formula. Again, we use this work to build the truth table below.

Example 18.

p	q	r	$(p \vee q)$	$((p \vee q) \wedge r)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	T	F	F
T	T	F	T	F
T	F	F	T	F
F	T	F	T	F
F	F	F	F	F

Remark 3. *A couple comments:*

1. *The number of rows of truth values in your table is equal to the number 2 raised the power which is the number of variables. Note that our last example had 3 variables, and $2^3 = 8$, which is the number of rows in the table.*

2. It does NOT MATTER in what order your fill in the truth values under the variables; all you need to do is make sure that you list all possiblilites (and don't repeat...)

Let's conclude with a final example, which will allow us to transition into the next lecture material for Wednesday.

Example 19. $((p \vee (\neg p)) \vee q)$ *is a formula.*

Both p and q are formulas by 1. of the definition. By 2., $(\neg p)$ is a formula. By 2. again, $(p \vee (\neg p))$ is a formula. Finally, by 2., we see that $((p \vee (\neg p)) \vee q)$ is a formula. And here's the truth table:

Example 20.

p	q	$(\neg p)$	$(p \vee (\neg p))$	$((p \vee (\neg p)) \vee q)$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	T	T

Remark 4. *Note that the above formula is true* regardless of the truth values of the propositional variables. *This is an example of a formula called a **tautology**, which we will discuss next lecture.*