Math 2150 Notes - June 29, 2022

Today, we will focus on truth and expressing uniqueness. There are no new definitions or techniques to learn here per se, but the problems will be more challenging, just a fair warning. Let's dive in with an example.

Example 1. *Determine the truth/falsity of the following, where the domain for all variables is the set of real numbers.*

1. $\exists x \exists y(x + y = 0)$,
2. $\forall x \exists y(x + y = 0)$,
3. $\exists x \forall y(x + y = 0)$, *and*
4. $\forall x \forall y(x + y = 0)$.[1]

Solution We deal with each of these in succession.

(1) The translation is, "There exist real numbers x and y such that $x + y = 0$." So the question is this: can you find a real number to substitute for x and a real number to substitute for y so that $x + y = 0$? Yes, of course. Let $x = 1$ and $y = -1$, for example. Hence this proposition is true.

(2) The translation is, "For every real number x, there exists a real number y such that $x + y = 0$." Notice that the sentence begins with the universally quantified variable x. Here's what to do (this is VERY IMPORTANT!): imagine reaching into the set of real numbers blindfolded. You pick out a real number x but you do NOT know what number is in your hand. Even though you don't know what x is, can you determine if what follows "For every real number x," is true? The question then becomes (this is what follows): does there exist a real number y such that $x+y = 0$? What if x happens to be 2? Then you could choose $y = -2$. What if x happens to be -4? Then you could choose $y = 4$. So this is some data to support an answer of yes, right? Now you don't know what x is, but if you choose $-x$ for y, then $x + y = x + -x = 0$, so you DO know there is such a real number y, even without knowing what x is. So this proposition is also true.

(3) The translation is, "There exists a real number x such that for all real numbers y, $x + y = 0$." Note that the quantifiers have now flipped, and this changes the meaning by leaps and bounds. Now the question becomes: can you find a real number x with the property that NO MATTER WHAT REAL NUMBER y YOU ADD TO x, you get $x + y = 0$? Is it true that no matter what real number y you add to 0, that $0 + y = 0$. No, of course not: if you add 1 to 0, you get 1, not 0. Is it true that no matter what real number y you add to 203, you get $203 + y = 0$? Again, of course not: if you add 1 to 203, you get 204, not zero. Hopefully, you can feel that no matter what real number x is, it is NOT TRUE that NO MATTER WHAT YOU ADD TO x, you get 0: indeed, if x is any real number, then if you add the real number $-x + 1$ to x, you get $x + -x + 1 = 1$, not zero. So there is NO real number x with the property that no matter what you add to x, you get 0. Thus this is a false proposition.

(4) The translation is, "For all real numbers x and y, $x + y = 0$." In other words, NO MATTER WHAT REAL NUMBERS x and y ARE, $x + y$ IS ALWAYS 0. This is OBVIOUSLY false: if $x = 1$ and $y = 3$, then $x + y$ is not equal to zero, so this is a false proposition. □

Let's look at another example.

Example 2. *Is $\forall x \exists y(xy = 1)$ true, if we let the domain for x and for y be the set of real numbers?*

Solution The translation is, "For every real number x, there exists a real number y such that $xy = 1$." So again, the idea is this: suppose you hold some real number x in your hand that you don't know. Can you find a real number y necessarily that yields $xy = 1$? Well, if $x = 2$, you could choose $y = \frac{1}{2}$; if $x = \frac{2}{3}$, you could choose $y = \frac{3}{2}$. But what if $x = 0$? Then there is no such y. BECAUSE THE ASSERTION IS FALSE

[1] Technically, "$x + y = 0$" is a two-place predicate and does not have parentheses. I inserted the parentheses into these statements for readability, but syntactically, this doesn't follow the definition I gave in the notes.

FOR AT LEAST ONE VALUE OF x, THE ASSERTION IS NOT TRUE FOR ALL VALUES OF x, and so this proposition is false. □

Next, let's discuss the task of expressing that there is exactly one of something.

Example 3. *Express that there is exactly one object, where the domain for whatever variables you choose is the collection of all objects.*

Solution The idea here is that we want to say that (a) there is an object x and (b) there is no OTHER object y that is different from x. One way to do this is as follows: $\exists x(\neg\exists y(y \neq x))$. This translates to "There exists an object x such that there does not exist an object y such that $y \neq x$." This does the job. Now consider applying DeMorgan: $\exists x(\neg\exists y(y \neq x)) \equiv \exists x\forall y(y = x)$, so the latter is another way to express this. □

Example 4. *Express that there are exactly two objects, where the domain for the variables is again the collection of all objects.*

Solution Here, we want to say that (a) there are objects x and y, (b) x and y are different (if they are the same, then we have one object, not two), and (c) there is no OTHER object z that is different from x and y. So we can say $\exists x\exists y(x \neq y \wedge (\neg\exists z(z \neq x \wedge z \neq y)))$. Again, we can "Demorganize" this formula to obtain $\exists x\exists y(x \neq y \wedge \forall z(z = x \vee z = y))$. □

Now let's utilize the above concepts in the context of other predicates.

Example 5. *Let $C(x, y)$ be "x and y have chatted online". Express "Bob has chatted with exactly one person online" if the domain for x and y is the set of all people.*

Solution Here we want to say that (a) Bob has chatted with some person x and (b) Bob has not chatted with any person y different from x: $\exists x(C(Bob, x) \wedge \neg\exists y(y \neq x \wedge C(Bob, y)))$. Note that the general idea was the same as in Example 48, but we now have to deal with an additional predicate. □

Example 6. *Let $C(x, y)$ be as above. Express "Bob and Sharon have chatted with the same people online".*

Solution What we are really saying here is that if Bob has chatted with any person x online, then Sharon also chatted with x online and if Sharon chatted with any person x online, then Bob also chatted with x online. In other words, Bob chatted with x if and only if Sharon chatted with x. So we can write (notice that "any" suggests a universal quantifier) $\forall x((C(Bob, x) \leftrightarrow C(Sharon, x))$. □
Let's look at yet another example.

Example 7. *Let $C(x, y)$ be as above. Express "Bob has chatted online with everyone that Sharon has chatted with online".*

Solution What we want to formalize is this: if Sharron has chatted with any person x online, then so has Bob: $\forall x(C(Sharon, x) \rightarrow C(Bob, x))$. □
Let's keep going with examples.

Example 8. *Determine if the following are true or false if the domain for all variables is the set of real numbers.*

1. $\exists x\exists y\exists z(x + y = z)$,
2. $\forall x\exists y\exists z(x + y = z)$,
3. $\exists x\forall y\exists z(x + y = z)$,
4. $\exists x\exists y\forall z(x + y = z)$.

Solution We deal with each in succession.

(1) Certainly there exist real numbers x, y, and z such that $x + y = z$: just take $x = 2$, $y = 3$, and $z = 5$, for example.

(2) Remeber what I said in my last set of notes: this formula is universally quantified out front, so we begin by reaching into the bag of real numbers and blindly pick a real number x whose value we don't know. Now the question we ask ourselves is this: "Do there exist real numbers y and z for which $x + y = z$?" If $x = 2$, then observe that if we choose $y = 0$ and $z = 2$, then we do in fact have $x + y = z$. If $x = 5$, then if we choose $y = 1$ and $z = 6$, we also have $x + y = z$. So it appears that the answer is that this statement is true. Note that if we let $y = 0$ and $z = x$, then we have $x + y = z$, and so we don't even need to know the value of x to see that the statement is indeed true.

(3) This one is a bit trickier. The method I am presenting to deal with it is a bit ad hoc, but it is important for you to "get your hands dirty" in order to get a feel for these kinds of problems, and I encourage you to consider employing these sort of "playing around" techniques when you get stuck. Suppose we let $x = 0$. Is it true that for all y, there exists z such that $0 + y = z$? In other words, is it true that for all real numbers y, there exists a real number z such that $y = z$? YES, of course: for any real number y, just choose $z = y$. So we saw that if we let $x = 0$, then $\forall y \exists z(x + y = z)$ is true. When you are trying to determine the truth or falsity of a formula that begins with an existential quantifier, you may consider letting the existentially quantified variable be some particular value and then ask yourself if with that value, is what follows true? Be aware that you may get one truth value for some values but a different truth value for others. For an existentially quantified formula (that is, a formula which begins w/ an existentially quantified variable) to be true, remember that it must simply be true for *at least one* value. Since we saw that our formula was true for $x = 0$, the entire formula is true.

(4) Now the question we ask ourselves is the following: can we find real numbers x and y such that for every real number z, we have $x + y = z$? Well, again, let's play with this a bit. Suppose we choose $x = 1$ and $y = 2$. Is it true that for every real number z, $1 + 2 = z$? No: if $z = 5$, then this statement is clearly false. Similarly, no matter what real number values you choose for x and y, it will not be hard for you to find a real value for z for which $x + y \neq z$. Indeed, if x and y are any real numbers, let $z = x + y + 1$. Then observe that $x + y \neq z$ in this case. □

Example 9. *Let $F(x, y)$ be "x and y are friends" and let $Y(x)$ be "x is a senior citizen". Express as formulas, where the domain for each variable is the set of all people.*

1. *Everyone has a senior citizen as a friend,*
2. *Bob is friends with every senior citizen,*
3. *Julie has exactly one senior citizen friend, and*
4. *Everyone is friends with him/herself.*

Solution Again, we deal with these successively.

(a) We want to express that for every person x, there is a senior citizen y such that x is friends with y: $\forall x \exists y(Y(y) \wedge F(x, y))$.

(b) We want to express that for every senior citizen y, Bob is friends with y: $\forall y(Y(y) \rightarrow F(Bob, y))$. Note that we are NOT expressing that Bob is friends with EVERYONE, only the senior citizens. This is the reason for the conditional statement.

(c) We want to express is that there is a senior citizen y such that Julie is friends with y and that there is no senior citizen z different from y such that Julie is friends with z:
$\exists y((Y(y) \wedge F(Julie, y)) \wedge (\neg \exists z((Y(z) \wedge z \neq y) \wedge F(Julie, z))))$.

(d) $\forall x F(x, x)$. □