Math 2150 Notes - July 18, 2022

To begin, I'm going to restate our list of assumptions as they will continue to play a role in our work.

Assumptions

1. basic algebra[1]
2. the sum, difference, product, and negatives of integers are integers,
3. every integer is either even or odd,
4. no integer is both even and odd,
5. $1 > 0$,
6. for all real numbers a, b, and c: if $a < b$, then $a + c < b + c$,
7. for all real numbers a, b, and c: if $a < b$ and $c > 0$, then $ac < bc$,
8. for all real numbers a, b, and c: if $a < b$ and $b < c$, then $a < c$, and
9. for all real numbers a, b, and c: exactly one of $a = b$, $a < b$, $b < a$ holds.
10. every rational number can be expressed in reduced form.
11. the product of two nonzero real numbers is nonzero.
12. if x is an integer and $x|1$, then $x = \pm 1$.

Today we continue with our treatment of proofs. Next up is learning how to prove an if and only if statement. Recall from propositional logic that $(p \leftrightarrow q)$ is called a *biconditional* statement and is read "p if and only if q". This formula is true precisely when p and q have the same truth values. Via a truth table, it is not hard to show that $(p \leftrightarrow q)$ is logically equivalent to $((p \to q) \wedge (q \to p))$. This leads to the next algorithm: proving and "if and only if" statement.

Definition 1 (Proving an if and only if statement)**.** *To prove a statement of the form* $\forall x_1 \cdots \forall x_n(P(x_1, \ldots, x_n) \leftrightarrow Q(x_1, \ldots, x_n))$*, do the following:*

1. *let* $x_1, \ldots, x_n$ *be arbitrary in their respective domains.*
2. *Prove* $P(x_1, \ldots, x_n) \to Q(x_1, \ldots, x)n)$*, and then*
3. *prove* $Q(x_1, \ldots, x_n) \to P(x_1, \ldots, x_n)$*.*

Let's check out an example.

Example 1. *Prove that for every integer* n*:* n *is even if and only if* $n + 1$ *is odd.*

Preformalization: $\forall n(n \text{ is even} \leftrightarrow n + 1 \text{ is odd})$.

Proof. Let n be an arbitrary integer. Assume first that n is even. We will prove that $n + 1$ is odd. Since n is even, $n = 2k$ for some integer k. Thus, adding 1 to both sides, $n + 1 = 2k + 1$, and so $n + 1$ is odd. Now assume that $n + 1$ is odd. We will prove that n is even. Since $n + 1$ is odd, $n + 1 = 2k + 1$ for some integer k. Subtracting 1 from both sides, we get $n = 2k$, and hence n is even. □

This was a particularly simple example. Let's look at one that is a bit more complicated.

[1] When you do basic algebra, you need NOT say "by assumption 1."; you can just do it in the course of a proof.

Example 2. *Prove that for every integer n, n is odd if and only if $3n^2+1$ is even.*

Preformalization: $\forall n(n$ is odd $\leftrightarrow 3n^2+1$ is even$)$.

Proof. Let n be an arbitrary integer. Assume first that n is odd. We must show that $3n^2+1$ is even. Since n is odd, we see that $n = 2k+1$ for some integer k. Hence $3n^2+1 = 3(2k+1)^2+1 = 3(4k^2+4k+1)+1 = 12k^2+12k+3+1 = 12k^2+12k+4 = 2(6k^2+6k+2)$. By assumption (2), we see that $6k^2+6k+2$ is an integer. Hence $3n^2+1$ is even. Next we must show that if $3n^2+1$ is even, then n is odd. This will not be easy to do via a direct proof, so let's use a proof by contraposition. Thus assume that n is not odd. We must prove that $3n^2+1$ is not even. By assumption (3), it follows that since n is not odd, n must be even. Thus $n = 2s$ for some integer s. Hence $3n^2+1 = 3(2s)^2+1 = 3(4s^2)+1 = 12s^2+1 = 2(6s^2)+1$. By assumption (2), $6s^2$ is an integer, and we have now shown that $3n^2+1$ is odd. Invoking assumption (4), $3n^2+1$ is not even. This completes the proof. $\square$

We now generalize this idea and show how to prove that a collection of statements $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are all logically equivalent. What this means is that either $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are ALL TRUE OR ALL FALSE. I will now give you the algorithm for proving that a collection of statements are logically equivalent, with a brief discussion to follow.

Definition 2 (Proving that a collection of propositions are equivalent)**.** *To prove that* $\forall x_1 \cdots \forall x_n \mathcal{S}_1 \leftrightarrow \mathcal{S}_2 \leftrightarrow \cdots \leftrightarrow \mathcal{S}_n$*, do the following:*

1. *let $x_1, \ldots, x_n$ be arbitrary members of their respective domains.*
2. *Prove $\mathcal{S}_1 \to \mathcal{S}_2$,*
3. *prove $\mathcal{S}_2 \to \mathcal{S}_3$,*
4. $\vdots$

n prove $\mathcal{S}_{n-1} \to \mathcal{S}_n$, and finally

n+1 prove that $\mathcal{S}_n \to \mathcal{S}_1$.

Here is the idea behind why this works: if you complete all of these steps, then if any one of the statements is true, then they ALL must be true. For suppose that you've completed these steps and you happen to know that $\mathcal{S}_2$ is true. Then because you have shown that $\mathcal{S}_2 \to \mathcal{S}_3$, it follows that $\mathcal{S}_3$ is true. Because you have shown that $\mathcal{S}_3 \to \mathcal{S}_4$, also $\mathcal{S}_4$ is true. Continuing, you get that $\mathcal{S}_2, \ldots, \mathcal{S}_n$ are all true. But you have also proved that $\mathcal{S}_n \to \mathcal{S}_1$, and so you know that $\mathcal{S}_1$ is true as well. Thus they are all true. It follows that all n statements are true or all n statements are false.
Before proceeding, let's revisit a definition given previously.

Definition 3. *A real number r is* ***rational*** *provided $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$. If r is a real number that is not rational, then we say that r is* ***irrational****.*

And now, a final example. First, I've added an assumption to the assumption list: assumption (11), which we invoke below.

Example 3. *Let x be a real number. Prove that the following are equivalent.*

1. *x is rational,*
2. *$x+1$ is rational,*
3. *$2x$ is rational.*

Proof. Let x be an arbitrary real number.

1. $\to$ 2.: assume that x is rational. We will prove that $x+1$ is also rational. Since x is rational, $x = \frac{a}{b}$ for some integers a and b, with $b \neq 0$. Thus $x + 1 = \frac{a}{b} + 1 = \frac{a+b}{b}$. By assumption (2), $a + b$ is an integer. As we already know that b is a nonzero integer, we deduce that $x + 1$ is rational, as desired.

2. $\to$ 3.: assume that $x+1$ is rational. We will prove that $2x$ is rational. Since $x+1$ is rational, $x+1 = \frac{a}{b}$ for some integers a and b with b nonzero. Thus, solving for x, we get $x = \frac{a-b}{b}$. Multiplying by 2, we see that $2x = \frac{2(a-b)}{b}$. Now, it follows by assumption (2) that $2(a-b)$ is an integer. As b is a nonzero integer, we see that $2x$ is also rational.

3. $\to$ 1.: assume that $2x$ is rational. We will prove that x is rational. Since $2x$ is rational, $2x = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Solving for x, we get $x = \frac{a}{2b}$. Now, $2b$ is an integer by assumption (2). Moreover, by assumption (11), $2b$ is nonzero. Hence x is rational, as desired. ☐

Our next proof technique is so-called "proof by cases."

Definition 4 (Proofs by Cases)**.** *To prove a statement $\forall x_1 \forall x_2 \cdots \forall x_n \mathcal{S}$ by cases, do the following:*

1. *Let $x_1, \ldots, x_n$ be arbitrary in their respective domains,*
2. *give an exhaustive treatment which proves $\mathcal{S}$ for *all possibilities* of values $x_1, \ldots, x_n$ can assume.*

Of course, item (2) above is a bit vague. Let's make this more concrete with a specific example.

Example 4. *Prove that for all integers n, $n^2 \geq n$.*

Preformalization: $\forall n n^2 \geq n$.

Proof. Let n be an arbitrary integer. We must prove that $n^2 \geq n$. There are three possibilities for n: either $n = 0$, $n > 0$, or $n < 0$. We prove that in each case, $n^2 \geq n$.

Case 1. $n = 0$. Then $n^2 = 0^2 = 0 \geq 0 = n$, and so $n^2 \geq n$ in this case.

Case 2. $n > 0$. Now, recall that n is an integer (a whole number or its negative). Thus since $n > 0$, we must have $n \geq 1$. Multiplying through by $n > 0$ (why can we do this?), we see that $n^2 \geq n$, which is what we wanted to prove.

Case 3. $n < 0$. Then by assumption (6), we see that $0 < -n$, that is, $-n > 0$. Multiplying both sides of $n < 0$ by $-n$ (by assumption (7)), we see that $-n^2 < 0$. Again, adding n^2 to both sides (assumption (6)), we have $0 < n^2$, that is, $n^2 > 0$. So we now have $n^2 > 0 > n$ (recall above that $n < 0$), and so by transitivity (assumption (8)), we get $n^2 > n$, and hence $n^2 \geq n$ in this case as well. This concludes the proof. ☐

For our next example, we shall need the definition of absolute value.

Definition 5. *Let x be a real number. Then the **absolute value** of x, denoted $|x|$, is defined as follows: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.*

Please take a moment to reflect on this definition and convince yourself that it conforms to your intuition for both positive and negative values of x. We now prove the following. Note that I am not proving all of the inequality assumptions below, but they can all be proven rigorously from the assumption list. In fact, after the next proof, I will do an example illustrating this.

Example 5. *Prove that for all real numbers x and y, we have $|xy| = |x||y|$.*

Proof. Let x and y be arbitrary real numbers. We consider several cases (the idea is that we consider cases which will allow us to drop the absolute value signs on x and y, respectively).

Case 1. $x \geq 0$ and $y \geq 0$: then $xy \geq 0$, and so $|xy| = xy = |x||y|$.

Case 2. $x \geq 0$ and $y < 0$. Then $xy \leq 0$, and so $|xy| = -xy = x(-y) = |x||y|$.

Case 3. $x < 0$ and $y \geq 0$. Then $xy \leq 0$, and so $|xy| = -xy = (-x)y = |x||y|$.

Case 4. $x < 0$ and $y < 0$. Then $xy > 0$, and so $|xy| = xy = (-x)(-y) = |x||y|$. This concludes the proof. □

Now let's give a couple of proofs which close some holes in the above argument.

Example 6. *Prove that for all real numbers x: if $x \leq 0$, then $|x| = -x$.*

Proof. (this will also be a proof by cases) Let x be an arbitrary real number. Assume that $x \leq 0$. We shall prove that $|x| = -x$. Toward this end, we consider two cases.

Case 1. $x = 0$. Then $|x| = 0 = -0 = -x$, and so we are done in this case.

Case 2. $x < 0$. Then $|x| = -x$, and we are done here as well. This completes the proof. □

Example 7. *Prove that for all real numbers x and y: if $x < 0$ and $y < 0$, then $xy > 0$.*

Proof. Let x and y be arbitrary real numbers. Assume that $x < 0$ and $y < 0$. We shall prove that $xy > 0$. By assumption (6), we see that since $y < 0$, we may add $-y$ to both sides to get $0 < -y$, that is, $-y > 0$. So now we know that $x < 0$ and $-y > 0$. By assumption (7), it follows that (multiplying through by $-y$) $x(-y) < 0(-y)$, that is, $-xy < 0$. Invoking assumption (6) again, we may add xy to both sides of $-xy < 0$ to obtain $0 < xy$, that is, $xy > 0$. □

We now introduce another definition.

Definition 6. *Let n be an integer. Then we say that n is a* ***perfect square*** *provided that $n = x^2$ for some integer x.*

Before proceeding, let's pause to do another proof by cases. I will include a preformalization this time.

Example 8. *For every real number x, we have $x^2 \geq 0$.*

Preformalization: $\forall x x^2 \geq 0$.

Proof. Let x be an arbitrary real number. We again consider cases.

Case 1. $x = 0$. Then $x^2 = 0^2 = 0$, and so $x^2 \geq 0$ in this case.

Case 2. $x > 0$. Then by assumption (7), we may multiply through by x to obtain $x^2 > 0 \cdot x$, that is, $x^2 > 0$. Hence in this case too, $x^2 \geq 0$.

Case 3. $x < 0$. By assumption (6), we may add $-x$ to both sides of the inequality to get $0 < -x$, that is, $-x > 0$. But now by assumption (7), we may multiply both sides of $x < 0$ by $-x$ to get $-x^2 < 0$. By assumption (6), we may add x^2 to both sides of this inequality to get $0 < x^2$, that is, $x^2 > 0$. This completes the proof. □

We can now give a proof by cases that for every integer n: if n is a perfect square, then the ones digit of n is either $0, 1, 4, 5, 6$, or 9.

Example 9. *Prove that for any perfect square n, the ones digit of n is either* 0, 1, 4, 5, 6, *or* 9.

Proof. Let n be an arbitrary integer, and assume that n is a perfect square. Then by definition, $n = m^2$ for some integer m. We consider cases determined by the ones digit of m.

Case 1. The ones digit of m is 0. Then the ones digit of $m^2 = n$ is also 0.

Case 2. The ones digit of m is 1. Then the ones digit of $m^2 = n$ is also 1.

Case 3. The ones digit of m is 2. Then the ones digit of $m^2 = n$ is 4.

Case 4. The ones digit of m is 3. Then the ones digit of m^2 is 9.

Case 5. The ones digit of m is 4. Then the ones digit of m^2 is 6 (since $4^2 = 16$; we carry the one, of course).

Case 6. The ones digit of m is 5. Then the ones digit of m^2 is also 5 (for the above reason).

Case 7. The ones digit of m is 6. Then the ones digit of m^2 is also 6.

Case 8. The ones digit of m is 7. Then the ones digit of m^2 is 9.

Case 9. The ones digit of m is 8. Then the ones digit of m^2 is 4.

Case 10. The ones digit of m is 9. Then the ones digit of m^2 is 1.

Thus is all cases, the ones digit of $n = m^2$ is either 0, 1, 4, 5, 6, or 9. □

We will conclude this set of notes with a proof involving perfect squares.

Example 10. *Prove that there do not exist* positive *and* consecutive *perfect squares.*

Proof. Suppose by way of contradiction that there exist positive integers a and b, both perfect squares, such that $b = a + 1$ (this is what consecutive means). Since a is a perfect square, $a = m^2$ for some integer m; since b is also a perfect square, $b = n^2$ for some integer n. Now, $b = a + 1$, and so we see that $n^2 = m^2 + 1$. Subtracting m^2 from both sides, we obtain $n^2 - m^2 = 1$. Factoring on the left, $(n + m)(n - m) = 1$. By assumption (2), both $n + m$ and $n - m$ are integers. The only pairs of integers that multiply to yield 1 are 1 and 1 and -1 and -1. Thus either $n + m = n - m = 1$ or $n + m = n - n = -1$.

Case 1. $n+m = n-m = 1$. Adding, we get $2n = 2$ and so $n = 1$. But then $m = 0$, and so $a = m^2 = 0^2 = 0$, contradicting that a is positive.

Case 2. $n + m = n - m = -1$. Adding, we get $2n = -2$, and so $n = -1$. But then again, this forces $m = 0$, and so as above, $a = 0$, contradicting that a is positive. □