MATH 3410 Autumn 2022 Lecture 7

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Absolute Value

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1. Assume that $x \geq 0$ and $y \geq 0$. We will prove that $xy \geq 0$. If $x = 0$ or $y = 0$, then $xy = 0$ (why?), and thus $xy \geq 0$.

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Let x and y be arbitrary real numbers.

1. Assume that $x \geq 0$ and $y \geq 0$. We will prove that $xy \geq 0$. If $x = 0$ or $y = 0$, then $xy = 0$ (why?), and thus $xy \geq 0$. Thus assume that x and y are nonzero.

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Let x and y be arbitrary real numbers.

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4. Assume that $x \leq 0$. We will show that $|x| = -x$. Since $x \leq 0$, either $x = 0$ or $x < 0$. If $x = 0$, then $x \geq 0$, and so by definition, $|x| = x = 0 = -0$ (why is it that $0 = -0$?).

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For all real numbers x *and* y*, we have* $|xy| = |x||y|$.

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Note first that if $0 \leq x < 1$, then $|x| = x < 1$. What about negative such values of x? Well, $|-\frac{1}{2}| = \frac{1}{2} < 1$ and $|-\frac{5}{6}| = \frac{5}{6} < 1$.

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For all real numbers x *and all positive real numbers* y*:* $|x| < y$ *if and only if* $-y < x < y$*.*

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To prove the forward implication, suppose by way of contradiction that there is a real number x and a positive real number y such that $|x| < y$, but $-y < x < y$ is false.

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To prove the forward implication, suppose by way of contradiction that there is a real number x and a positive real number y such that $|x| < y$, but $-y < x < y$ is false. Then by Trichotomy, either $x \leq -y$ or $y \leq x$.

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Case 1: $x \leq -y$. Then $x \leq -y < 0$ (why?), and so $x < 0$ (why?). By definition of absolute value, $|x| = -x$. Recall above that $|x| < y$, and thus $-x < y$. By Additive Invariance, we may add $x - y$ to both sides of $-x < y$ to get $-y < x$, that is, $x > -y$.

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Case 1: $x \leq -y$. Then $x \leq -y < 0$ (why?), and so $x < 0$ (why?). By definition of absolute value, $|x| = -x$. Recall above that $|x| < y$, and thus $-x < y$. By Additive Invariance, we may add $x - y$ to both sides of $-x < y$ to get $-y < x$, that is, $x > -y$. But (by the case we are in) $x \leq -y$, and this contradicts Trichotomy.
Case 2: $y \leq x$: then $0 < y \leq x$, and so $0 < x$ (why?); in other words, $x > 0$.

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To prove the forward implication, suppose by way of contradiction that there is a real number x and a positive real number y such that $|x| < y$, but $-y < x < y$ is false. Then by Trichotomy, either $x \leq -y$ or $y \leq x$.
Case 1: $x \leq -y$. Then $x \leq -y < 0$ (why?), and so $x < 0$ (why?). By definition of absolute value, $|x| = -x$. Recall above that $|x| < y$, and thus $-x < y$. By Additive Invariance, we may add $x - y$ to both sides of $-x < y$ to get $-y < x$, that is, $x > -y$. But (by the case we are in) $x \leq -y$, and this contradicts Trichotomy.
Case 2: $y \leq x$: then $0 < y \leq x$, and so $0 < x$ (why?); in other words, $x > 0$. Again, recall above that $|x| < y$.

Absolute Value

Theorem

For all real numbers x and all positive real numbers y: $|x| < y$ if and only if $-y < x < y$.

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The following hold:

1. $|1 + 2| = |3| = 3 \leq |1| + |2|$,
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The following hold:

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2. $|-1+-2| = |-3| = 3 \leq |-1| + |-2|$,
3. $|-3+5| = |2| = 2 \leq |-3| + |5|$, and
4. $|3+-7| = |-4| = 4 \leq |3| + |-7|$.

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Theorem (Triangle Inequality)

For all real numbers x *and* y*, we have* $|x + y| \leq |x| + |y|$.

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Let x and y be arbitrary real numbers.

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Case 1: $x \geq 0$ and $y \geq 0$. Then $x + y \geq 0$ (why?), and so $|x + y| = x + y = |x| + |y|$; hence $|x + y| \leq |x| + |y|$.

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Case 2: $x < 0$ and $y < 0$. Then $x + y < 0$ (why?), and so $|x + y| = -(x + y) = -x + -y = |x| + |y|$; thus $|x + y| \leq |x| + |y|$ (note that you proved in hw#2 that $-(x + y) = -x + -y$).

Absolute Value

Theorem (Triangle Inequality)

For all real numbers x and y, we have $|x+y| \leq |x|+|y|$.

Proof.

Let x and y be arbitrary real numbers. We will proceed by cases.
Case 1: $x \geq 0$ and $y \geq 0$. Then $x+y \geq 0$ (why?), and so $|x+y| = x+y = |x|+|y|$; hence $|x+y| \leq |x|+|y|$.
Case 2: $x < 0$ and $y < 0$. Then $x+y < 0$ (why?), and so $|x+y| = -(x+y) = -x+-y = |x|+|y|$; thus $|x+y| \leq |x|+|y|$ (note that you proved in hw#2 that $-(x+y) = -x+-y$).
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Case 3: $x \geq 0$ and $y < 0$. This also comes with two cases: either $x+y \leq 0$ or $x+y < 0$.

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Let x and y be arbitrary real numbers. We will proceed by cases.
Case 1: $x \geq 0$ and $y \geq 0$. Then $x + y \geq 0$ (why?), and so $|x+y| = x + y = |x| + |y|$; hence $|x+y| \leq |x| + |y|$.
Case 2: $x < 0$ and $y < 0$. Then $x + y < 0$ (why?), and so $|x+y| = -(x+y) = -x + -y = |x| + |y|$; thus $|x+y| \leq |x| + |y|$ (note that you proved in hw#2 that $-(x+y) = -x + -y$).
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Case 3i: $x + y \geq 0$.

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Case 1: $x \geq 0$ and $y \geq 0$. Then $x+y \geq 0$ (why?), and so $|x+y| = x+y = |x|+|y|$; hence $|x+y| \leq |x|+|y|$.
Case 2: $x < 0$ and $y < 0$. Then $x+y < 0$ (why?), and so $|x+y| = -(x+y) = -x+-y = |x|+|y|$; thus $|x+y| \leq |x|+|y|$ (note that you proved in hw#2 that $-(x+y) = -x+-y$).
Case 3: $x \geq 0$ and $y < 0$. This also comes with two cases: either $x+y \leq 0$ or $x+y < 0$.
Case 3i: $x+y \geq 0$. Then $|x+y| = x+y = |x|+y \leq |x|+|y|$ (since $y < 0$, $y < |y|$, and so by Additive Invariance, $|x|+y < |x|+|y|$; hence $|x|+y \leq |x|+|y|$).

□

Absolute Value

Continued.

Case 3ii: $x + y < 0$. Then
$|x + y| = -(x + y) = -x + -y = -x + |y| \leq |x| + |y|$ (since $x \geq 0$, we have $-x \leq |x|$ (why?), and so $-x + |y| \leq |x| + |y|$ (why? Note that this is NOT from Additive Invariance).

Absolute Value

Continued.

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Case 4: $x < 0$ and $y \geq 0$.

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Case 4: $x < 0$ and $y \geq 0$. Then $y \geq 0$ and $x < 0$.

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Case 4: $x < 0$ and $y \geq 0$. Then $y \geq 0$ and $x < 0$.by Case 3, we see that $|y + x| \leq |y| + |x|$.

Absolute Value

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$|x + y| = -(x + y) = -x + -y = -x + |y| \leq |x| + |y|$ (since $x \geq 0$, we have $-x \leq |x|$ (why?), and so $-x + |y| \leq |x| + |y|$ (why? Note that this is NOT from Additive Invariance).
Case 4: $x < 0$ and $y \geq 0$. Then $y \geq 0$ and $x < 0$.by Case 3, we see that $|y + x| \leq |y| + |x|$. Because $+$ is commutative, this becomes $|x + y| \leq |x| + |y|$, and the proof is complete. □

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Let $S = \{1, 2, 3\}$.

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Now let $S = \{0, 1, 2, 3, 4, 5, \ldots\}$. Is there a real number r such that $s \leq r$ for ALL $s \in S$?

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Completeness

Definition

Let S be a subset of $\mathbb{R}$, and let $r \in \mathbb{R}$.

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Solution.

By definition of "negative", if $s \in S$, then $s \leq 0$ (s is STRICTLY less than 0, but this doesn't matter).

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By definition of "negative", if $s \in S$, then $s \leq 0$ (s is STRICTLY less than 0, but this doesn't matter). Thus 0 is an upper bound for S. But ANY non-negative real number is also an upper bound for S, since if r is non-negative, then for any negative real number s, we have $s \leq 0 \leq r$, and hence $s \leq r$. □

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Suppose that S is any *nonempty* subset of the real numbers. If there exists an upper found for S in $\mathbb{R}$, then there is a least upper bound for S in $\mathbb{R}$.

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If S is a subset of $\mathbb{R}$ for which there exists an upper bound for S, then we say that S is **bounded above**.

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What does it MEAN for the empty set to be bounded above? It means that there exists some real number r such that $s \leq r$ for all $s \in \varnothing$. (again, this is where it is ESSENTIAL that you have a facility with predicate logic, or again, you will likely fail the course) So is this true? What is the negation of this statement? It is: for all real numbers r, there exists some $s \in \varnothing$ such that $s > r$. But THIS IS FALSE BECAUSE IF IT WERE TRUE, THERE WOULD EXIST AN ELEMENT OF THE EMPTY SET!!! The upshot: $\varnothing$ IS bounded above. □

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So you see: there is reason why the axiom asserts only that every NONEMPTY subset of $\mathbb{R}$ that has an upper bound has a least upper bound.