Math 3410 Assignment 2 Solutions

Throughout, justify each step in your proof by appealing to an axioms OR to an example/proposition/theorem proved in the notes.

(0)[I deducted 20 points if you didn't list these] List all the embedded words from the 8/29 and 8/31 lectures (there are six).

Solution. cat, dog, fox, car, truck, tractor. □

(1)[10 pts] Prove that for every nonzero real number x, we have $\frac{1}{\frac{1}{x}} = x$. Preformalize first.

Preformalization: $\forall x(\frac{1}{\frac{1}{x}} = x)$ (where the domain for x is the set of *nonzero* real numbers)

Proof. Let x be an arbitrary *nonzero* real number. By the Multiplicative Inverse Axiom and the Commutativity of Multiplication Axiom, we have $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$. Thus by Example 5 of the 8/31 notes, $x = \frac{1}{\frac{1}{x}}$, and we are done (yes, it is that short and sweet). □

(2)[10 pts] Prove that forall real numbers x and y, we have $-(x + y) = -x + -y$. Preformalize first.

Preformalization: $\forall x \forall y[-(x + y) = -x + -y]$

Proof. Let x and y be arbitrary real numbers. Then observe that $x + y + (-x + -y) = x + -x + y + -y = 0 + 0 = 0$ (we are invoking the Commutativity of Addition Axiom, the Additive Identity Axiom, and the Additive Inverse Axiom). Thus by Example 3 of the 8/31 notes, $-x+-y = -(x+y)$, as was to be shown. □

(3)[10 pts] Prove that for all nonzero real numbers x and y, we have $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}$. Preformalize first. Hint: consider $x \cdot y \cdot \frac{1}{x} \cdot \frac{1}{y}$, and apply a result in the notes right before the natural numbers introduction.

Preformalization: $\forall x \forall y(\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy})$

Proof. Let x and y be arbitrary nonzero real numbers. Then note that $(x \cdot y) \cdot (\frac{1}{x} \cdot \frac{1}{y})$[1] $= x \cdot \frac{1}{x} \cdot y \cdot \frac{1}{y}$ (by Commutativity of Multiplication Axiom) $= 1 \cdot 1 = 1$ (by Multiplicative Identity and Inverse Axioms). Thus by Example 5 of the 8/31 notes, we deduce that $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{xy}$. □

[1] You don't need the parentheses here; I'm adding them for emphasis so you can see how we can apply Example 5 at the end.

(4)[10 pts] We may now define division as follows: if a is a real number and b is a nonzero real number, then define $\frac{a}{b} = a \cdot \frac{1}{b}$. Prove that for all real numbers a and b and all nonzero real numbers c, we have $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$. Preformalize first. Hint: the proof is NOT long; you really just need the definition and the Distributive Axiom.

Preformalization: $\forall a \forall b \forall c (\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c})$ (where the domain for a and b is the set of real numbers and the domain for c is the set of nonzero real numbers)

Proof. Let a and b be arbitrary real numbers and let c be an arbitrary nonzero real number. Then by definition, we see that $\frac{a+b}{c} = (a+b) \cdot \frac{1}{c} = a \cdot \frac{1}{c} + b \cdot \frac{1}{c}$ (by the Distributive Axiom and Commutativity of Multiplication Axiom) $= \frac{a}{c} + \frac{b}{c}$ (by definition). □

(5)[10 pts] Prove that for all $m, n \in \mathbb{N}$, we have $m + n \in \mathbb{N}$. You do not need to preformalize first. Hint: let $m \in \mathbb{N}$ be arbitrary. Now prove by induction on n that for all natural numbers n, $m + n \in \mathbb{N}$.

Proof. Let m be an arbitrary natural number. We will prove by induction on n that $m + n \in \mathbb{N}$ for every natural number n.

(i) (base case) We must check that $m + 0 \in \mathbb{N}$. This is easy: $m + 0 = m$ (Additive Identity Axiom), and $m \in \mathbb{N}$ by our assumption above.

(ii) (inductive step) Let $n \in \mathbb{N}$ be arbitrary, and assume that $m + n \in \mathbb{N}$. We must show that $m + n + 1 \in \mathbb{N}$ (recall my comments about not needing parentheses because of associativity). We must show that $m + n + 1$ is a member of every inductive subset of $\mathbb{R}$, since by definition, $\mathbb{N}$ is the intersection of all such subsets. Let I be an arbitrary inductive subset of $\mathbb{R}$. Since $m + n \in \mathbb{N}$, $m + n \in I$. Now, because I is inductive, $m + n + 1 \in I$. This concludes the proof. □

(6)[10 pts] Prove for all $m, n \in \mathbb{N}$ $m \cdot n \in \mathbb{N}$. Hint: proceed as in (5). In the inductive step, you will use the distributive property as well as (5).

Proof. Let $m \in \mathbb{N}$ be arbitrary. We will prove by induction that $m \cdot n \in \mathbb{N}$ for all $n \in \mathbb{N}$ by induction on n.

(i) (base case) We must check that $m \cdot 0 \in \mathbb{N}$. By the Multiply by Zero Proposition, $m \cdot 0 = 0 \in \mathbb{N}$ (recall from the notes that $\mathbb{N}$ is inductive, and hence $0 \in \mathbb{N}$).

(ii) (inductive step) Let $n \in \mathbb{N}$ be arbitrary and assume that $m \cdot n \in \mathbb{N}$. Then note that $m \cdot (n+1) = m \cdot n + m \cdot 1 = m \cdot n + m$ (by the Distributive Axiom, the Multiplicative Identity Axiom, and the Commutativity of Multiplication Axiom). Recall from (5) that the sum of any two natural numbers is a natural number. Since $m \cdot n$ and m are natural numbers, so is $m \cdot n + m$, and we are done. □