Math 3410 Assignment 3 Solutions

You will not need (nor should you) to use induction for ANY of the following problems.

(0) List all the embedded words from the 9/7 (there are three).

Solution. house, condo, property. □

(1)[10 pts] Prove that for all real numbers a and c and all nonzero real numbers b and d, we have $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Note that for the quantity on the right side of = to be defined, we need to know that $bd \neq 0$. You will need to justify this in the course of your proof (hint: Zero Product Property from lecture notes). You may use problem #3 from hw#2 here if it helps you (and of course, you don't need to reprove it - you can just quote it).

Proof. Let a and c be real numbers and let b and d be nonzero real numbers. The contrapositive of the Zero Product Property says that if x and y are nonzero real numbers, then $xy \neq 0$. Thus $bd \neq 0$. Further, $\frac{a}{b} \cdot \frac{c}{d} = a \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d} = ac \cdot \frac{1}{b} \cdot \frac{1}{d}$ by Commutativity of Multiplication. By problem 3 of hw#2, $\frac{1}{b} \cdot \frac{1}{d} = \frac{1}{bd}$, and upon substituting this in the right side of the final equation above, we get $\frac{a}{b} \cdot \frac{c}{d} = ac \cdot \frac{1}{bd} = \frac{ac}{bd}$. □

(2)[10 pts] Prove that for all real numbers a and c and all nonzero real numbers b and d, we have $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. This proves that the "getting a common denominator and adding" algorithm is correct. Similar remarks to those given in (1) also apply here.

Proof. Let a and c be real numbers and let b and d be nonzero real numbers. By the (contrapositive of) the Zero Product Property, $bd \neq 0$. Furthermore, $\frac{ad+bc}{bd} = \frac{ad}{bd} + \frac{bc}{bd}$ (you proved this in #4 of hw#2) $= ad \cdot \frac{1}{bd} + bc \cdot \frac{1}{bd} = ad \cdot \frac{1}{b} \cdot \frac{1}{d} + bc \cdot \frac{1}{b} \cdot \frac{1}{d}$ (here, we are applying #3 of hw#2) $= ad \cdot \frac{1}{d} \cdot \frac{1}{b} + b \cdot \frac{1}{b} \cdot c \cdot \frac{1}{d}$ (why?) $= a \cdot 1 \cdot \frac{1}{b} + 1 \cdot c \cdot \frac{1}{d}$ (why?) $= a \cdot \frac{1}{b} + c \cdot \frac{1}{d}$ (why?) $= \frac{a}{b} + \frac{c}{d}$. □

(3) Recall the definition of the set $\mathbb{Q}$ of rational number: $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$.

(a)[5 pts] Prove that the set $\mathbb{Q}$ of rational numbers is closed under multiplication, that is, prove that for all rational numbers x and y, we have $x \cdot y \in \mathbb{Q}$ as well. You will need to use some of the results from the notes on $\mathbb{Z}$ here as well as (1) on the previous page. The proof is NOT long.

Proof. Let $x, y \in \mathbb{Q}$ be arbitrary. Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. By problem 1, $x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. We have seen that $bd \neq 0$. Since the set $\mathbb{Z}$ is closed under multiplication (proved in the notes), we now have $ac \in \mathbb{Z}$ and $bd \in \mathbb{Z} \backslash \{0\}$. Hence by definition, $xy \in \mathbb{Q}$. $\square$

(b)[5 pts] Prove that $\mathbb{Q}$ is closed under addition, that is, prove that for all rational numbers x and y, we have $x + y \in \mathbb{Q}$. Similar remarks apply here as in (a).

Proof. Let $x, y \in \mathbb{Q}$. Then $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. Thus $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ by (2). As above, $bd \neq 0$. Since $\mathbb{Z}$ is closed under multiplication *and* addition (proved in the notes), we see that $ad + bc \in \mathbb{Z}$ and $bd \in \mathbb{Z} \backslash \{0\}$, proving that $x + y \in \mathbb{Q}$. $\square$

(4)[10 pts] Prove that for all real numbers x and all nonzero real numbers y, we have $-(\frac{x}{y}) = \frac{-x}{y}$.

Proof. Recall from the notes (find it!) that for all real numbers a and b: if $a + b = 0$, then $b = -a$. We will use that result here. Let x be a real number and let y be a nonzero real number. Then $\frac{x}{y} + \frac{-x}{y} =$ (from hw#2) $\frac{x+-x}{y} = \frac{0}{y} = 0 \cdot \frac{1}{y} = 0$ by the Multiply by Zero Proposition. Thus $\frac{-x}{y} = -(\frac{x}{y})$, and the proof is complete. $\square$

(5)[5 pts] Use (4) and the notes on the integers to prove that $\mathbb{Q}$ is closed under negatives, that is, for any $x \in \mathbb{Q}$, also $-x \in \mathbb{Q}$. The proof is not long.

Proof. Let $x \in \mathbb{Q}$ be arbitrary. Then $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$. By (4), we have $-x = -\frac{a}{b} = \frac{-a}{b}$. Since $\mathbb{Z}$ is closed under negatives (notes!), we see that $-a \in \mathbb{Z}$. This proves that $-x \in \mathbb{Q}$. $\square$

(6)[5 pts] Use (4) and (5) to prove that $\mathbb{Q}$ is closed under subtraction, that is, for any $x, y \in \mathbb{Q}$, also $x - y \in \mathbb{Q}$ (remember that I defined subtraction in the notes). The proof is also not long.

Proof. Let $x, y \in \mathbb{Q}$. Then from (5), $-y \in \mathbb{Q}$. From (3b), $x + -y = x - y \in \mathbb{Q}$. $\square$