MATH 3410 Autumn 2022 Lecture 6

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This following hold:

1. 1 is less than 5,
2. -2 is less than 0,
3. -7 is less than -1,
4. 6 is not less than 6.

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Axiom (Trichotomy)

For any real numbers a and b, **exactly one** of the following holds:

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Suppose by way of contradiction that there exists a real number r such that $r < r$. We know that $r = r$ (everything is equal to itself). But then we have both $r = r$ and $r < r$, and this contradicts Trichotomy. □

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Let a, b, and c be real numbers, and assume that $a \leq b$ and $b \leq c$. We will show that $a \leq c$.To do this, we consider four cases.

Case 1: $a = b$ and $b = c$.

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Case 2: $a = b$ and $b < c$. Then (substituting a for b in $b < c$) we get $a < c$, and so $a \leq c$ in this case too.

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Case 4: $a < b$ and $b < c$. Then by the Transitivity Axiom, $a < c$, and hence $a \leq c$. □

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Let r be an arbitrary real number. We will show that $r^2 \geq 0$.From the Trichotomy Axiom, either $r = 0$, $r > 0$, or $r < 0$. We consider these cases separately.

Case 1: $r = 0$.

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Case 1: $r = 0$. Then $r^2 = 0^2 = 0$ (by the Multiply by Zero Proposition), and so by definition, $r^2 \neq 0$ in this case.

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Case 3: $r < 0$.

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Case 3: $r < 0$. By Additive Invariance, we can add $-r$ to both sides to get $r + -r < 0 + -r = -r$. So $0 < -r$, that is, $-r > 0$. By Case 2, $(-r)^2 \geq 0$. But $(-r)^2 = (-r)(-r) = r^2$ by Negative Product Cancellation, and so $r^2 \geq 0$ here too.

Order Axioms

Corollary

$1 > 0$.

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We will finish up for today with two more proofs to help with intuition and to assist with the homework.

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We will finish up for today with two more proofs to help with intuition and to assist with the homework. Some of your homework problems will be similar to what I have done in the lecture notes.

Example

For all real number a and b and c: if $a \leq b$ and $c \geq 0$, then $ac \leq bc$.

Order Axioms

Proof.

Let a, b, and c be real numbers.

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Order Axioms

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Let a, b, and c be real numbers. Assume that $a \leq b$ and $c \geq 0$. We will show that $ac \leq bc$.

Case 1: $a = b$. The $ac = bc$, and $ac \leq bc$.

Case 2: $a < b$.

Order Axioms

Proof.

Let a, b, and c be real numbers. Assume that $a \leq b$ and $c \geq 0$. We will show that $ac \leq bc$.

Case 1: $a = b$. The $ac = bc$, and $ac \leq bc$.

Case 2: $a < b$. Now, $c \geq 0$, so either $c = 0$ or $c > 0$.

Order Axioms

Proof.

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Case 1: $a = b$. The $ac = bc$, and $ac \leq bc$.

Case 2: $a < b$. Now, $c \geq 0$, so either $c = 0$ or $c > 0$. If $c = 0$, then $ac = a0 = 0 = b0 = bc$, and $ac \leq bc$.

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Proof.

Let a, b, and c be real numbers. Assume that $a \leq b$ and $c \geq 0$. We will show that $ac \leq bc$.

Case 1: $a = b$. The $ac = bc$, and $ac \leq bc$.

Case 2: $a < b$. Now, $c \geq 0$, so either $c = 0$ or $c > 0$. If $c = 0$, then $ac = a0 = 0 = b0 = bc$, and $ac \leq bc$. Now assume that $c > 0$.

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Proof.

Let a, b, and c be real numbers. Assume that $a \leq b$ and $c \geq 0$. We will show that $ac \leq bc$.

Case 1: $a = b$. The $ac = bc$, and $ac \leq bc$.

Case 2: $a < b$. Now, $c \geq 0$, so either $c = 0$ or $c > 0$. If $c = 0$, then $ac = a0 = 0 = b0 = bc$, and $ac \leq bc$. Now assume that $c > 0$. Then we have $a < b$ and $c > 0$; by Multiplicative Invariance, $ac < bc$, and again, we conclude that $ac \leq bc$. □

Order Axioms

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For all real numbers x: if $x > 0$, then $\frac{1}{x} > 0$.

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Suppose by way of contradiction that there exists a real number x such that $x > 0$, but it is not the case that $\frac{1}{x}$ is greater than 0.

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Proof.

Suppose by way of contradiction that there exists a real number x such that $x > 0$, but it is not the case that $\frac{1}{x}$ is greater than 0. By the Trichotomy Axiom, it follows that either $\frac{1}{x} = 0$ or $\frac{1}{x} < 0$.

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