MATH 3410 Lecture 13

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More Limit Laws

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Let (x_n), (y_n), and (z_n) be real sequences which satisfy the above hypotheses. We will prove that $(y_n) \to 0$.

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Let $\epsilon > 0$ be arbitrary.

General Limits

Let's conclude the lecture for today with a proof of another limit.

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Prove that the sequence $(\frac{n}{n+1})$ converges to 1.

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Let $\epsilon > 0$ be arbitrary. We must produce a natural number k such that for all natural numbers $n \geq k$, we have $|\frac{n}{n+1} - 1| < \epsilon$.

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Let $\epsilon > 0$ be arbitrary. We must produce a natural number k such that for all natural numbers $n \geq k$, we have $|\frac{n}{n+1} - 1| < \epsilon$. Getting a common denominator for the term inside the absolute value signs, this is equivalent to producing a natural number k such that for every natural number $n \geq k$, we have $|\frac{-1}{n+1}| < \epsilon$, that is, $\frac{1}{n+1} < \epsilon$.

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