Math 3410 Lecture 14

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General Limits of Sequences

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But now $x_n < \frac{L+M}{2} < x_n$, and so by Transitivity, $x_n < x_n$, a contradiction. □

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Let (x_n), (y_n) *be sequences, and let* $L, M \in \mathbb{R}$. *If* $(x_n) \to L$ *and* $(y_n) \to M$, *then* $(x_n + y_n) \to L + M$.

Proof.

Suppose that $(x_n) \to L$ and $(y_n) \to M$, where $L, M \in \mathbb{R}$. Then by definition, $(x_n - L) \to 0$ and $(y_n - M) \to 0$. It follows from Limit Law 2 that $(x_n - L + y_n - M) \to 0$. Written another way, we get $(x_n + y_n - (L + M)) \to 0$. But this *MEANS* (look back at the definition of convergence of a general sequence. Really. Do it.) that $(x_n + y_n) \to L + M$.

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Theorem (General Limit Law 3)

If $(x_n) \to L$ *and* $(y_n) \to M$*, then* $(x_n y_n) \to LM$ *(for any real sequences* (x_n) *and* (y_n) *and any real numbers* L *and* M*)*

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Suppose that $(x_n) \to L$, $x_n \neq 0$ *for all* $n \in \mathbb{N}$, *and* $L \neq 0$.

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I'm only going to informally talk through the proof; I'm not going to give the formal argument. The reason for this is because I want to try to impart the intuition that precedes the proof. We know that $(x_n) \to L$, which means that x_n gets arbitrarily close to L as n gets arbitrarily large. We want for $\frac{1}{x_n}$ to get arbitrarily close to $\frac{1}{L}$ as n gets arbitrarily large. Said in another way, we know that $x_n - L$ gets arbitrarily close to 0 as n gets arbitrarily large, and we want to show that $\frac{1}{x_n} - \frac{1}{L}$ gets arbitrarily close to 0 as n gets arbitrarily large. Note that, getting a common denominator, $\frac{1}{x_n} - \frac{1}{L} = \frac{L - x_n}{L x_n}$. Note that $L - x_n = -(x_n - L)$.

General Limits of Sequences

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