Math 3410 Assignment 6 Solutions

(0) Tell me the embedded words from the September 26 and September 28 lectures.

(1)[3 pts (completion)] The Principle of Strong Induction assertes the following: suppose that $T \subseteq \mathbb{N}$ satisfies the following conditions:

1. $0 \in T$, and
2. for every natural number n, if $m \in T$ for all natural numbers $m \leq n$, then $n+1 \in T$.

Conclusion: $T = \mathbb{N}$.

Prove the Principle of Strong Induction. Hint: let $S = \{n \in \mathbb{N} \colon m \in T$ for all natural numbers $m \leq n\}$, where T is given as above. Show that S is inductive using the (regular) Principle of Mathematical Induction, and conclude that $S = \mathbb{N}$. Now argue from this that $T = \mathbb{N}$. This problem basically proves itself IF you are proficient with logic and understanding what these words above mean. The proof requires no clever tricks, just understanding of semantics.

Proof. Let T be a subset of $\mathbb{N}$ which satisfies 1. and 2. above. Now let $S = \{n \in \mathbb{N} \colon m \in T$ for all natural numbers $m \leq n\}$. We first show that S is inductive.

(i) (base case) We must show that $0 \in S$, that is, we must show that $m \in T$ for all natural numbers $m \leq 0$. You proved in the last hw that $n \geq 0$ for every natural number n. Thus showing that $0 \in S$ is equivalent to showing that $0 \in T$, which is true by 1.

(ii) (inductive step) Let $n \in \mathbb{N}$, and assume that $n \in S$. We must show that $n+1 \in S$. Since $n \in S$, this means that $m \in T$ for every natural number $m \leq n$. By 2., $n+1 \in T$. So now $m \in T$ for every natural number $m \leq n+1$ (can you see that there is no natural number x such that $n < x < n+1$?). By definition of S, this yields that $n+1 \in S$.

By the Principal of Mathematical Induction, it follows that $S = \mathbb{N}$. We need to prove that $T = \mathbb{N}$. By assumption, $T \subseteq \mathbb{N}$. It remains to show that $\mathbb{N} \subseteq T$. Let $n \in \mathbb{N}$ be arbitrary. Since $S = \mathbb{N}$, it follows that $n \in S$. By definition of S, this means that $m \in T$ for all natural number m such that $m \leq n$. Since $n \leq n$, we see that $n \in T$, and the proof is complete. □

(2)[3 pts (completion)] The Well-Ordering Theorem asserts that every nonempty subset of $\mathbb{N}$ has a least element. Prove the Well-Ordering Theorem using the Principle of Strong Induction. Hint: suppose by way of contradiction that there exists a nonempty subet S of $\mathbb{N}$ with NO least element, and let $T = \{n \in \mathbb{N} \colon n \notin S\}$. Prove that T satisfies items 1. and 2., and thus $T = \mathbb{N}$. A contradiction immediately arises...

Proof. Suppose by way of contradiction that S is a nonempty subset of $\mathbb{N}$ with no least element. Now let $T = \{n \in \mathbb{N} \colon n \notin S\}$. We will show that T satisfies 1. and 2. of (1). Note first that $0 \notin S$, lest 0 be the least element of S; thus $0 \in T$. Now let $n \in \mathbb{N}$, and assume that $0, \ldots, n \in T$. This means that $0, \ldots, n$ are not in S. It follows that $n+1 \notin S$, since if $n+1 \in S$, $n+1$ would be the least element of S ($n+1$ is least because we have that NONE of $0, 1, \ldots, n$ are in S, and so if $n+1 \in S$, then $n+1$ would be the LEAST member of S). Thus $n+1 \in T$. By the Principle of Strong Induction, $T = \mathbb{N}$. But then NO member of $\mathbb{N}$ is in S, and so $S = \varnothing$, a contradiction. □

(3)[10 pts] Prove that for all positive real numbers a and b: if $a^2 < b^2$, then $a < b$. Prove this FROM THE AXIOMS AND NOTES - do not perform any operations on inequalities without justification. My suggestion: proof by contradiction, or proof by contraposition.

Proof. Let a and b be positive real numbers. Assume that it is not the case that $a < b$. We will show that it is not the case that $a^2 < b^2$. Since $a < b$ is false, $b \leq a$ holds by trichotomy. If $b = a$, then we can square both sides to get $b^2 = a^2$, and thus (by trichotomy) $a^2 < b^2$ is false. Now suppose that $b < a$. By Mult. Inv., we may multiply through by b to get $b^2 < ab$. Multiplying $b < a$ through by a, we get $ab < a^2$ (again, this is justified by Mult. Inv.). So we have $b^2 < ab < a^2$. By Transitivity, $b^2 < a^2$; trichotomy implies that $a^2 < b^2$ is false, and the proof is complete. □

(4)[10 pts] Prove that for all natural numbers n, we have $(-1)^n = 1$ or $(-1)^n = -1$. Prove this by (regular) induction on $\mathbb{N}$. There is NO reason for you to mention 'even' or 'odd' here, so don't do it.

Proof. We proceed by induction, showing that for every natural number n, either $(-1)^n = 1$ or $(-1)^n = -1$.

(i) (base case) $(-1)^0 = 1$, so this proves the base case.

(ii) (inductive step) Let $n \in \mathbb{N}$, and assume that $(-1)^n = 1$ or $(-1)^n = -1$. We will prove that the same holds with the n replaced with $n + 1$. We consider two cases.

Case 1: $(-1)^n = 1$. Then simply multiply both sides by -1 to get $(-1)^{n+1} = 1 \cdot -1 = -1$.

Case 2: $(-1)^n = -1$. Again, we multiply both sides by -1 to get $(-1)^{n+1} = (-1)^2 = 1$ by Multiplicative Negative Cancellation, and this completes the proof. □

(5)[10 pts] Prove that the sequence $(\frac{(-1)^n}{n+1})$ converges to 0.YOU CAN AND SHOULD USE THE NOTES HERE WHERE I PROVED THAT $(\frac{1}{n+1})$ converges to 0, as well as (4). All this requires is essentially 'gluing' the results of (4) and the final example of the notes together. You do NOT have to do much work outside of this. But be CAREFUL to ask yourself 'why' after every sentence you write; do NOT just copy and paste without thinking.

Proof. Let $\epsilon > 0$. Since $(\frac{1}{n+1}) \to 0$, there is some $k \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \geq k$, then $|\frac{1}{n+1}| = \frac{1}{n+1} < \epsilon$. Now let $n \geq k$. Then from (4), we see that $|\frac{(-1)^n}{n+1}| = |\frac{1}{n+1}|$ or $|\frac{(-1)^n}{n+1}| = |\frac{-1}{n+1}| = |-\frac{1}{n+1}|$ (Why? This is a special case of an old hw problem!). In any case, we have $|\frac{(-1)^n}{n+1}| = \frac{1}{n+1} < \epsilon$, and this proves that $(\frac{(-1)^n}{n+1}) \to 0$. □

(6)[10 pts] Prove that for every real number r, we have $||r|| = |r|$ (that is, the absolute value of the absolute value of r is the same thing as the absolute value of r). Prove EVERYTHING rigorously here by appealing to the axioms, the notes, and/or old hw.

Proof. Let $r \in \mathbb{R}$ be arbitrary. To prove that $||r|| = |r|$, all we need to do is show that $|r| \geq 0$ (WHY?). If $r \geq 0$, then $|r| = r \geq 0$. Now suppose that $r < 0$. Then by Additive Invariance (adding $-r$ to both sides), we get $0 < -r$. Also, by definition, $|r| = -r > 0$, and so $|r| \geq 0$ in this case as well. □

(7)[10 pts] Let (r_n) be a sequence of real numbers. Prove that $(r_n) \to 0$ if and only if $(|r_n|) \to 0$. This more or less proves itself if you know what everything means. Use (6).

Proof. Let (r_n) be a sequence of real numbers. Assume first that $(r_n) \to 0$. We will show that $(|r_n|) \to 0$. Toward this end, let $\epsilon > 0$ be arbitrary. We need to show that there is some $k \in \mathbb{N}$ such that for every natural number $n \geq k$, we have $||r_n|| < \epsilon$. Since $(r_n) \to 0$, there is $k \in \mathbb{N}$ such that for every natural number $n \geq k$, we have $|r_n| < \epsilon$. By (6), for $n \geq k$, we have $||r_n|| = |r_n| < \epsilon$, showing that $(|r_n| \to 0)$. Now assume that $(|r_n|) \to 0$. Let $\epsilon > 0$ be arbitrary. Then there is $k \in \mathbb{N}$ such that for all natural numbers $n \geq k$, we have $||r_n|| < \epsilon$. As above, for any such n, $|r_n| = ||r_n||$, and so $|r_n| < \epsilon$, proving that $(r_n) \to 0$. $\square$