Math 3410 Assignment 7 Solutions

(0) Give me the 6 embedded words (total) from the 10/10 and 10/12 lectures.

(1)[10 pts] Prove that the sequence $(\frac{n}{n+1})$ does not converge to 0.

Proof. Let $\epsilon = \frac{1}{4}$, and let $k \in \mathbb{N}$ be arbitrary. Let $n = \max(k, 1)$. Note that $n \geq k$. It suffices to show that $|\frac{n}{n+1}| \geq \frac{1}{4}$, that is, $\frac{n}{n+1} \geq \frac{1}{4}$. Note that this inequality is equivalent to $4n \geq n+1$ (why), which is equivalent to $3n \geq 1$. Since $n = \max(k, 1)$, we see that $n \geq 1$. Thus $3n \geq n \geq 1$, and hence $3n \geq 1$, and we are done. □

(2)[10 pts] Use Limit Law 1 and Limit Law 2 to prove that for all sequences (x_n) and (y_n): if $(x_n) \to 0$ and $(y_n) \to 0$, then $(x_n - y_n) \to 0$. Note: you can go about it the long, torturous way as I did in the notes, or, if you can see how to apply the two limit laws, you can do the proof in a couple of lines and avoid epsilons and k and n entirely...

Proof. Let (x_n) and (y_n) be sequences that both converge to 0. By Limit Law 1, $(-1y_n) = (-y_n) \to 0$. Thus by Limit Law 2, $(x_n - y_n) = (x_n + -y_n) \to 0$. □

(3)[10 pts] Prove that the sequence $(\frac{3}{n+4}) \to 0$.

Proof. Let $\epsilon > 0$ be arbitrary. By the Archimedean Property, there is a positive integer k such that $\frac{1}{k} < \frac{\epsilon}{3}$. Now let $n \geq k$ be a natural number. Then $n + 4 > k$, and thus $\frac{1}{n+4} < \frac{1}{k}$ (remember this from Test 1?). So now $\frac{1}{n+4} < \frac{1}{k} < \frac{\epsilon}{3}$. By Transitivity, $\frac{1}{n+4} < \frac{\epsilon}{3}$. Multiplying through by 3, we get $\frac{3}{n+4} < \epsilon$, and the proof is complete. □

(4)[5 pts] Give an example of sequences (x_n) and (y_n) such that $(x_n) \to 0$, $(y_n) \to 0$, $y_n \neq 0$ for every natural number n, and $(\frac{x_n}{y_n}) \to 0$. You do NOT need to prove that your (x_n) and (y_n) sequences converge to 0 (though they need to), but you need to PROVE that $(\frac{x_n}{y_n}) \to 0$.

Solution. Let $x_n = 0$ for all n and let $y_n = \frac{1}{n+1}$. Then we have seen that both $(x_n) \to 0$ and $(y_n) \to 0$, and clearly $y_n \neq 0$ for all $n \in \mathbb{N}$. Observe that $\frac{x_n}{y_n} = 0$ for all $n \in \mathbb{N}$, and so by Test 1, $\frac{x_n}{y_n} \to 0$. □

(5)[5 pts] Give an example of sequences (x_n) and (y_n) such that $(x_n) \to 0$, $(y_n) \to 0$, $y_n \neq 0$ for every natural number n, and $(\frac{x_n}{y_n})$ does NOT converge to 0. Again, you need not prove that (x_n) and (y_n) converge to 0, but you need to prove that $(\frac{x_n}{y_n})$ does NOT converge to 0.

Solution. Let $x_n = y_n = \frac{1}{n+1}$. Then as we have seen, both $(x_n) \to 0$ and $(y_n) \to 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$. Moreover, $\frac{x_n}{y_n} = 1$ for all natural numbers n. Let $\epsilon = 1$, and let $k \in \mathbb{N}$ be arbitrary. Now let $n = k$. Then observe that $|\frac{x_n}{y_n}| = |1| = 1 \geq 1$, and this shows that $(\frac{x_n}{y_n})$ does not converge to 0. □

(6)[10 pts] Let S be a subset of $\mathbb{R}$. Recall that S is bounded below if there is some $r \in \mathbb{R}$ such that $r \leq s$ for all $s \in S$. Similarly, S is bounded above if there is some $r \in \mathbb{R}$ such that $s \leq r$ for every $s \in S$. Let us say that S is **bounded** provided S is bounded below and bounded above. Prove that S is bounded if and only if there exists a real number r such that $|s| \leq r$ for all $s \in S$.

Proof. Let S be a subset of $\mathbb{R}$. Suppose first that there is some $r \in \mathbb{R}$ such that $|s| \leq r$ for all $s \in S$. If S is empty, S is clearly bounded below and above (since any real number is both a lower and an upper bound for S). So suppose that S is nonempty, and let $s \in S$. Then $0 \leq |s| \leq r$, and it follows that $r \geq 0$. Let $s \in S$ and suppose that $s \neq \pm r$. Then $|s| < r$. By the first theorem of the 9/14 notes, $-r < s < r$. It follows that for any $s \in S$, we have $-r \leq s \leq r$, and so S is bounded. Suppose now that S is bounded. By assumption, there are real numbers r_1 and r_2 such that $r_1 \leq s \leq r_2$ for all $s \in S$. Next, observe that $r_2 \leq |r_2| \leq |r_1| + |r_2|$. Similarly, $-(|r_1| + |r_2|) \leq -|r_1| \leq r_1$. It follows that for any $s \in S$, we have $-(|r_1| + r_2|) \leq s \leq |r_1| + |r_2|$, and thus $|s| \leq |r_1| + |r_2|$, concluding the proof. □