MATH 2150 Lecture 12

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University of Colorado
Colorado Springs

Assumption List

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1. basic algebra[1]

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To prove a statement of the form $\forall x_1 \forall x_2 \cdots \forall x_n (P(x_1, x_2, \ldots, x_n) \to Q(x_1, x_2, \ldots, x_n))$ by contraposition, do the following:

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(c) for all real numbers a, b, and c: if $a < b$ and $c > 0$, then $ac < bc$,

(d) for all real numbers a, b, and c: if $a < b$ and $b < c$, then $a < c$, and

(e) for all real numbers a and b: exactly one of $a = b$, $a < b$, $b < a$ holds.

Before proving the next example, recall that, for real numbers x and y, $x > y$ means $y < x$.

Proof by Contraposition

Next, I will introduce the basic properties of the usual order $<$ on the real line (these have also been added to our list of assumptions). IN ANY HOMEWORK PROBLEMS, PLEASE REFER TO THE NUMBERS IN THE ASSUMPTION LIST IN YOUR PROOFS, AND NOT TO THE LETTERS BELOW.

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Before proving the next example, recall that, for real numbers x and y, $x > y$ means $y < x$.

Proof by Contraposition

Example

Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

Proof by Contraposition

Example

Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

Preformalization: $\forall a \forall b \forall c((a < b \wedge c < 0) \rightarrow ac > bc)$.

Proof by Contraposition

Example

Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

Preformalization: $\forall a \forall b \forall c((a < b \land c < 0) \to ac > bc)$. We will use a direct proof this time.

Proof by Contraposition

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Proof.

Let a, b, and c be arbitrary real numbers.

Proof by Contraposition

Example

Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

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Proof.

Let a, b, and c be arbitrary real numbers. Assume that $a < b$ and that $c < 0$.

Proof by Contraposition

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Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

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Proof.

Let a, b, and c be arbitrary real numbers. Assume that $a < b$ and that $c < 0$. We must prove that $ac > bc$.

Proof by Contraposition

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Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

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Proof.

Let a, b, and c be arbitrary real numbers. Assume that $a < b$ and that $c < 0$. We must prove that $ac > bc$. By assumption (6) ((b) above), $c + -c < 0 + -c$, that is $0 < -c$.

Proof by Contraposition

Example

Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

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Let a, b, and c be arbitrary real numbers. Assume that $a < b$ and that $c < 0$. We must prove that $ac > bc$. By assumption (6) ((b) above), $c + -c < 0 + -c$, that is $0 < -c$. But this means that $-c > 0$.

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Proof by Contraposition

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Prove that for all real numbers a, b, and c: if $a < b$ and $c < 0$, then $ac > bc$.

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Proof by Contraposition

Next, for real numbers x and y, we define $x \leq y$ to mean "$x < y$ or $x = y$" and $x \geq y$ to mean "$x > y$ or $x = y$".

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Example

For all real numbers a, b, and c: if $a \leq b$, then $a + c \leq b + c$.

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Example

For all real numbers a, b, and c: if $a \leq b$, then $a + c \leq b + c$.

Preformalization: $\forall a \forall b \forall c (a \leq b \to a + c \leq b + c)$.

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Let a, b, and c be arbitrary real numbers.

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Preformalization: $\forall a \forall b \forall c (a \leq b \rightarrow a + c \leq b + c)$.

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Let a, b, and c be arbitrary real numbers. Assume it is not the case that $a + c \leq b + c$.

Proof by Contraposition

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Proof.

Let a, b, and c be arbitrary real numbers. Assume it is not the case that $a + c \leq b + c$. We must show that it is false that $a \leq b$.

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For all real numbers a, b, and c: if $a \leq b$, then $a + c \leq b + c$.

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Proof.

Let a, b, and c be arbitrary real numbers. Assume it is not the case that $a + c \leq b + c$. We must show that it is false that $a \leq b$. Since it is false that $a + c \leq b + c$, it is false that either $a + c < b + c$ or $a + c = b + c$.

Proof by Contraposition

Next, for real numbers x and y, we define $x \leq y$ to mean "$x < y$ or $x = y$" and $x \geq y$ to mean "$x > y$ or $x = y$". We will use another proof by contraposition for our final example.

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Let a, b, and c be arbitrary real numbers. Assume it is not the case that $a + c \leq b + c$. We must show that it is false that $a \leq b$. Since it is false that $a + c \leq b + c$, it is false that either $a + c < b + c$ or $a + c = b + c$. By assumption (9) above ((e) above), we deduce that $b + c < a + c$.

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