Math 2150 Lecture 14

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Proofs of if and only if statements

Assumptions

1. basic algebra[1]
2. the sum, difference, product, and negatives of integers are integers,
3. every integer is either even or odd,
4. no integer is both even and odd,
5. $1 > 0$,
6. for all real numbers a, b, and c: if $a < b$, then $a + c < b + c$,
7. for all real numbers a, b, and c: if $a < b$ and $c > 0$, then $ac < bc$,
8. for all real numbers a, b, and c: if $a < b$ and $b < c$, then $a < c$, and
9. for all real numbers a, b, and c: exactly one of $a = b$, $a < b$, $b < a$ holds.
10. every rational number can be expressed in reduced form.
11. the product of two nonzero real numbers is nonzero.
12. if x is an integer and $x|1$, then $x = \pm 1$.

[1] When you do basic algebra, you need NOT say "by assumption 1."; you can just do it in the course of a proof.

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Definition (Proving an if and only if statement)

To prove a statement of the form $\forall x_1 \cdots \forall x_n (P(x_1, \ldots, x_n) \leftrightarrow Q(x_1, \ldots, x_n))$, do the following:

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We now generalize this idea and show how to prove that a collection of statements $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are all logically equivalent. What this means is that either $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are ALL TRUE OR ALL FALSE. I will now give you the algorithm for proving that a collection of statements are logically equivalent, with a brief discussion to follow.

Definition (Proving that a collection of propositions are equivalent)

To prove that $\forall x_1 \cdots \forall x_n \mathcal{S}_1 \leftrightarrow \mathcal{S}_2 \leftrightarrow \cdots \leftrightarrow \mathcal{S}_n$, do the following:

1. let $x_1, \ldots, x_n$ be arbitrary members of their respective domains.
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Let x be an arbitrary real number.

1. $\rightarrow$ 2.: assume that x is rational. We will prove that $x+1$ is also rational. Since x is rational, $x=\frac{a}{b}$ for some integers a and b, with $b\neq 0$. Thus $x+1=\frac{a}{b}+1=\frac{a+b}{b}$.

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Let x be an arbitrary real number.

1. $\rightarrow$ 2.: assume that x is rational. We will prove that $x + 1$ is also rational. Since x is rational, $x = \frac{a}{b}$ for some integers a and b, with $b \neq 0$. Thus $x + 1 = \frac{a}{b} + 1 = \frac{a+b}{b}$. By assumption (2), $a + b$ is an integer.

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Proof.

Let x be an arbitrary real number.

1. $\rightarrow$ 2.: assume that x is rational. We will prove that $x+1$ is also rational. Since x is rational, $x = \frac{a}{b}$ for some integers a and b, with $b \neq 0$. Thus $x + 1 = \frac{a}{b} + 1 = \frac{a+b}{b}$. By assumption (2), $a + b$ is an integer. As we already know that b is a nonzero integer, we deduce that $x + 1$ is rational, as desired. □

Proofs of if and only if statements

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Proof.

2. $\rightarrow$ 3.: assume that $x + 1$ is rational. We will prove that $2x$ is rational. Since $x + 1$ is rational, $x + 1 = \frac{a}{b}$ for some integers a and b with b nonzero.

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3. $\rightarrow$ 1.: assume that $2x$ is rational. We will prove that x is rational. Since $2x$ is rational, $2x = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Solving for x, we get $x = \frac{a}{2b}$.

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2. $\to$ 3.: assume that $x+1$ is rational. We will prove that $2x$ is rational. Since $x+1$ is rational, $x+1=\frac{a}{b}$ for some integers a and b with b nonzero. Thus, solving for x, we get $x=\frac{a-b}{b}$. Multiplying by 2, we see that $2x=\frac{2(a-b)}{b}$. Now, it follows by assumption (2) that $2(a-b)$ is an integer. As b is a nonzero integer, we see that $2x$ is also rational.

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