Math 3410 Lecture 16

Greg Oman

University of Colorado
Colorado Springs

Cauchy Sequences

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Theorem (Monotonic Convergence Theorem)

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Case 1: (x_n) has but finitely many peaks. This means that there exists a natural number m such that x_n is NOT a peak for any $n > m$.

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Every real sequence has a monotonic subsequence, which is also bounded (why?), and hence converges by the Monotonic Convergence Theorem. □

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