Math 2150 Assignment 8 Solutions

(0) Give me the embedded words from the 10/17 and 10/19 lectures.

(1)[10 pts] Use a proof by contradiction to prove that for all integers n: if n^4 is even, then n is even. Preformalize both the statement to be proved AND the negation.

Preformalization: $\forall n(n^4 \text{ is even} \to n \text{ is even})$; Negation: $\exists n(n^4 \text{ is even} \land n \text{ is not even})$

Proof. Assume by way of contradiction that there exists an integer n such that n^4 is even but n is not even. By assumption (3), n is odd. Thus $n = 2k+1$ for some integer k. Raising both sides to the 4th power, we get $n^4 = (4k^2+4k+1)^2 = 16k^4+16k^3+4k^2+16k^3+16k^2+4k+4k^2+4k+1 = 16k^4+32k^3+24k^2+8k+1 = 2(8k^4+16k^3+12k^2+4k)+1$. Assumption (2) implies that $8k^4+16k^3+12k^2+4k$ is an integer, and thus n^4 is odd. However, n^4 is even, and we have a contradiction to assumption (4). □

(2)[10 pts] Use a proof by contradiction to prove that $\sqrt[4]{2}$ (the 4th root of 2) is irrational. You will need the result of (1) above. Try to mimic what I did in the notes with the square root of 2 (but you'll need to make a few modifications in this setting). You do NOT need to preformalize here.

Proof. Suppose by way of contradiction that $\sqrt[4]{2}$ is rational; by assumption (11), we see that $\sqrt[4]{2} = \frac{a}{b}$ for some integers a and b with $b \neq 0$ and $\frac{a}{b}$ is reduced. Now raise both sides to the 4th power to get $2 = \frac{a^4}{b^4}$. Clearling the fraction, $2b^4 = a^4$. By assumption (2), b^4 is an integer, and so a^4 is even. By (1), a is even. Thus $a = 2k$ for some integer k. Substituting into $2b^4 = a^4$, we get $2b^4 = 16k^4$. Dividing by 2, $b^4 = 8k^4 = 2(4k^4)$. By assumption (2), $4k^4$ is an integer, and thus b^4 is even. By (1), b is even. But now 2 is a factor of both a and b, contradicting that $\frac{a}{b}$ is reduced. □

(3)[5 pts[1]] Use a proof by contradiction to prove that for all *positive* real numbers x and y: if $x^2 < y^2$, then $x < y$. Please be careful to appeal to the axioms here to justify your work. Preformalize the statement to be proved AND the negation.

Preformalization: $\forall x \forall y (x^2 < y^2 \to x < y)$; negation: $\exists x \exists y (x^2 < y^2 \wedge x \geq y)$

Proof. Suppose by way of contradiction that there exist positive real numbers x and y such that $x^2 < y^2$ and $x \geq y$. Since $x \geq y$, either $x = y$ or $x > y$.

Case 1: $x = y$. Then $x^2 = y^2$, contradicting that $x^2 < y^2$ (this is a contradiction to assumption (9)).

Case 2: $x > y$. This means that $y < x$. By assumption (7) (recall that y is positive) we may multiply both sides of $y < x$ by y to get $y^2 < xy$. Now, since $y < x$ and x is positive, we may also multiply both sides of $y < x$ by x to get $xy < x^2$. So now we have $y^2 < xy < x^2$. By transitivity, $y^2 < x^2$. But this contradicts $x^2 < y^2$. □

(4)[10 pts] Use a proof by contradiction to prove that for all real numbers x and y: if $xy = 0$, then $x = 0$ or $y = 0$. Preformalize the statement to be proved AND the negation.

Preformalization: $\forall x \forall y (xy = 0 \to (x = 0 \vee y = 0))$; negation: $\exists x \exists y (xy = 0 \wedge (x \neq 0 \wedge y \neq 0))$.

Proof. Assume by way of contradiction that there exist real numbers x and y such that $xy = 0$ but $x \neq 0$ and $y \neq 0$. Since $y \neq 0$, we may divide both sides of $xy = 0$ by y to get $x = 0$, contradicting that $x \neq 0$. □

[1] Technically, you are using cases to prove this statement, and since I hadn't formally covered proof by cases when I assigned this hw, I am only grading the set-up of the problem

(5)[not graded] Prove that for all integers x and y: $x + y$ is even if and only if x and y are both even or both odd. Preformalize first.

Preformalization: $\forall x \forall y (x + y$ is even$\leftrightarrow$ x and y have the same parity)

Sketch of Proof. Let x and y be arbigtrary integers. Assume (contraposition) that x and y do not have the same parity. Then by assumptions (3) and (4), one of x, y is even and the other is odd. It is then similar to previous problems to show that $x + y$ is odd, thus by assumption (4), not even. Conversely, suppose that x and y are both even or both odd. Then consider cases, where for the first case, x and y are even and in the second, x and y are odd, and then show that $x + y$ is even in both cases. □

(6)[15 pts] Prove that the following are equivalent for any real number r (no preformalization is necessary):

1. $r + 1$ is irrational.
2. $r - 1$ is irrational.
3. $\frac{r}{3}$ is irrational.

I *strongly* suggest using either a proof by contraposition or a proof by contradiction for each implication.

Proof. Let r be an arbitrary real number.

1. $\rightarrow$ 2.: assume that $r - 1$ is not irrational. We will show that $r + 1$ is not irrational. Since $r - 1$ is not irrational, $r - 1$ is rational. Thus $r - 1 = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Then $r + 1 = \frac{a}{b} + 2 = \frac{a+2b}{b}$. By assumption (2), $a + 2b$ is an integer, and thus $r + 1$ is rational, so not irrational.

2. $\rightarrow$ 3.: assume that $\frac{r}{3}$ is not irrational. We will show that $r - 1$ is not irrational. Since $\frac{r}{3}$ is not irrational, $\frac{r}{3}$ is rational. Thus $\frac{r}{3} = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Thus $r = \frac{3a}{b}$, and also $r - 1 = \frac{3a-b}{b}$. By assumption (2), $3a - b$ is an integer, and so $r - 1$ is rational, thus not irrational.

3. $\rightarrow$ 1.: Assume that $r + 1$ is not irrational. We will show that $\frac{r}{3}$ is not irrational. Since $r + 1$ is not irrational. $r + 1$ is rational. Thus $r + 1 = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Hence $r = \frac{a}{b} - 1 = \frac{a-b}{b}$. Hence $\frac{r}{3} = \frac{a-b}{3b}$. By assumption (11), $3b \neq 0$. Further, by assumption (2), $a - b$ and $3b$ are integers. Thus $\frac{r}{3}$ is rational, so not irrational. □