Math 2150 Assignment 9 Solutions

(0) Give me the embedded words from the 10/24 and 10/26 lectures.

(1)[10 pts] Use a proof by cases to prove that for all real numbers x, y, and z: if $x < y$ and $y \leq z$, then $x < z$. Hint: You will want to use assuption (8) in your proof. Note that assumption (8) states that if $x < y$ and $y < z$, then $x < z$. It says nothing about 'less than or equal to' inequalities. Preformalize first.

Preformalization: $\forall x \forall y \forall z((x < y \wedge y \leq z) \rightarrow x < z)$

Proof. Let x, y, and z be arbitrary real numbers. Assume that $x < y$ and $y \leq z$. We will show that $x < z$. Since $y \leq z$, either $y = z$ or $y < z$.

Case 1: $y = z$. Then since $x < y$, it follows that $x < z$, and we are done.

Case 2: $y < z$. Then $x < y$ and $y < z$. So by assumption (8), $x < z$ and we are done. □

(2)[10 pts] Use a proof by cases to prove that for all real numbers x, y, and z: if $x \geq y$ and $y \geq z$, then $x \geq z$. Note that assumption (8) will be used here as well. Also note that assumption (8) applies ONLY to strict 'less than' inequalities, not greater than inequalities or greater than or equal to inequalities. You need to use the definition of $\geq$ and cases to make sure that assumption (8) applies. Preformalize first.

Preformalization: $\forall x \forall y \forall z((x \geq y \wedge y \geq z) \rightarrow x \geq z)$.

Proof. Let x, y, and z be arbitrary real numbers. Assume that $x \geq y$ and $y \geq z$. We will show that $x \geq z$. We know that $x = y$ or $x > y$ and we know that $y = z$ or $y > z$. This yields 4 cases.

Case 1: $x = y$ and $y = z$. Then $x = z$ and so $x \geq z$.

Case 2: $x = y$ and $y > z$. Then $x > z$, and thus $x \geq z$.

Case 3: $x > y$ and $y = z$. Then $x > z$, so $x \geq z$.

Case 4: $x > y$ and $y > z$. Then $z < y$ and $y < x$. By assumption (8), $z < x$, that is $x > z$. Thus $x \geq z$. □

(3)[10 pts] Use a proof by cases to prove that for all real numbers x, we have $|x|^2 = x^2$. Preformalize first.

Preformalization: $\forall x(|x|^2 = x^2)$.

Proof. Let x be an arbitrary realy number. By assumption (9), either $x \geq 0$ or $x < 0$.

Case 1: $x \geq 0$. Then $|x|^2 = x^2$, since $|x| = x$ in this case.

Case 2: $x < 0$. Then $|x|^2 = (-x)^2 = (-x)(-x) = x^2$, and we are done. □

(4)[10 pts] Prove that if x and y are perfect squares and $y = x + 3$, then $x = 1$ and $y = 4$. You may assume that the only integer factors of 3 are $-1, 1, 3$, and -3. You do NOT need to preformalize here.

Proof. Let x and y be perfect squares and assume that $y = x + 3$. Then $y = a^2$ and $x = b^2$ for some integers a and b. Substituting, $a^2 = b^2 + 3$, that is, $a^2 - b^2 = 3$. Factoring, we get $(a + b)(a - b) = 3$. Thus either $a + b = 3$, $a - b = 1$, or $a + b = 1$, $a - b = 3$, or $a + b = -3$, $a - b = -1$, or $a + b = -1$ and $a - b = -3$.

Case 1: $a + b = 3$ and $a - b = 1$. Solving for a and b, $a = 2$ and $b = 1$. Thus $x = 1$ and $y = 4$, as desired.

Case 2: $a + b = 1$ and $a - b = 3$. Solving for a and b, $a = 2$ and $b = 1$, and the rest proceeds as above.

Case 3: $a + b = -3$ and $a - b = -1$. Solving for a and b, $a = -2$ and $b = -1$, and the rest proceeds as above.

Case 4: analogous. □

(5)[5 pts] Prove that there exists a real number x such that for all real numbers y, $xy - y = 0$. Preformalize first.

Preformalization: $\exists x \forall y (xy - y = 0)$.

Proof. Let $x = 1$, and now let y be an arbitrary real number. Then $xy - y = 1y - y = y - y = 0$. □

(6)[10 pts] Disprove that there exists a real number x such that $x + 1 < x$. Preformalize the statement to be disproved.

Preformalization: $\exists x(x + 1 < x)$

Disproof. Suppose by way of contradiction that there exists a real number x such that $x + 1 < x$. By assumption (6), we may add $-x$ to both sides of the previous inequality to get $1 < 0$, a contradiction to the fact that $1 > 0$. $\square$

(7)[5 pts] Disprove that for all real numbers x and y, $x^2 \leq y$. Preformalize the statement to be disproved.

Preformalization: $\forall x \forall y(x^2 \leq y)$

Disproof. We need to prove the negation, namely, that there exists a real number x and a real number y such that $x^2 > y$. Let $x = 1$ and $y = 0$. $\square$