Math 2150

Test 2 Solutions

(1)(a)[10 pts] Give a direct proof that for all integers x and y, if x is even and y is odd, then $x+3y$ is odd. Preformalize first.

Preformalization: $\forall x \forall y((x \text{ is even} \wedge y \text{ is odd}) \rightarrow x+3y \text{ is odd})$

Proof. Let x and y be arbitrary integers. Assume that x is even and y is odd. We will prove that $x+3y$ is odd. Since x is even, $x = 2m$ for some integer m; since y is odd, $y = 2n+1$ for some integer n. Thus $x+3y = 2m+3(2n+1) = 2m+6n+3 = 2m+6n+2+1 = 2(m+3n+1)+1$. By assumption 2, $m+3n+1$ is an integer, and so $x+3y$ is odd. □

(1)(b)[10 pts] Prove by contraposition that for any real number r, if r^2 is irrational, then r is irrational. Preformalize first.

Preformalization: $\forall r(r^2 \text{ is irrational} \rightarrow r \text{ is irrational})$

Proof. Let r be an arbitrary real number. Assume that r is not irrational. We will show that r^2 is not irrational. Since r is not irrational, r is rational. Thus $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$. Thus $r^2 = \frac{a^2}{b^2}$. By assumption 2, a^2 and b^2 are integers. By assumption 11, $b^2 \neq 0$. Thus r^2 is rational, so not irrational. □

(2)(a)[10 pts] Recall that if x and y are integers, then x is a **factor** of y if there is some integer z such that $xz = y$. If x is a factor of y, we denote this by $x|y$ (we discussed this in the notes/hw). Prove the following using a direct proof: for all integers x and y: if x is a factor of y, then x^2 is a factor of y^2. Preformalize first.

Preformalization: $\forall x \forall y (x|y \rightarrow x^2|y^2)$

Proof. Let x and y be integers. Assume that $x|y$. We will show that $x^2|y^2$. Since $x|y$, there is an integer z such that $xz = y$. Squaring both sides, we get $x^2z^2 = y^2$. By assumption 2, z^2 is an integer, and thus $x^2|y^2$. □

(2)(b)[10 pts] Prove the following by contradiction: for all nonzero real numbers x and y, if $xy > 0$, then x and y are both positive or both negative. Hint: you will want to use this fact: (*) if a and b are real numbers such that $a > 0$ and $b < 0$, then $ab < 0$. You do NOT need to prove this (I proved this in the notes). Aside from this, quote the assumptions from the assumption list to justify the steps in your proof. Preformalize the statement to be proved along with its negation.

I did NOT grade the preformalization on this problem - just wanted you to think it through first.

Proof. Assume by way of contradiction that there exist nonzero real numbers x and y such that $xy > 0$, but x and y are not both positive and not both negative. Then by assumption (9), one of x and y is positive and the other is negative.

Case 1: $x > 0$ and $y < 0$. Then by (*), $xy < 0$, and this contradicts $xy > 0$.

Case 2: $x < 0$ and $y > 0$. Then by (*), $xy < 0$, and we have a contradiction as above. □

(3)(a)[4 pts per implication] Prove that the following. No preformalization is necessary.

For every integer n, the following are equivalent.

1. n^2 is even.
2. n^3 is even.
3. n is even.

Hint: you can prove that 1. implies 2. using a direct proof.

Proof. Let n be an arbitrary integer.

1.→2.: Assume that n^2 is even. Then $n^2 = 2x$ for some integer x. Hence $n^3 = 2(xn)$. By assumption (2), xn is an integer, and so n^3 is even.

2.→3.: Assume that n is not even. Then by assumption (3), n is odd. So $n = 2x + 1$ for some integer x. Thus $n^3 = (4x^2+4x+1)(2x+1) = 8x^3+4x^2+8x^2+4x+2x+1 = 8x^3+12x^2+6x+1 = 2(4x^3+6x^2+3x)+1$. By assumption (2), $4x^2+6x^2+3x$ is an integer, and so n^3 is odd.

3.→1.: Assume that n is even. Then $n = 2x$ for some integer x. Thus $n^2 = 2(2x^2)$. By assumption (2), $2x^2$ is an integer, and so n^2 is even. □

(3)(b)[8 pts] Use a proof by cases to show that for all positive real numbers x and y: if $y \leq x$, then $\frac{1}{x} \leq \frac{1}{y}$. You may use the fact that if z is any positive real number, then $\frac{1}{z}$ is also positive. No preformalization necessary here.

Proof. Let x and y be positive real numbers. Assume that $y \leq x$. We will show that $\frac{1}{x} \leq \frac{1}{y}$.

Case 1: $y = x$. Then $\frac{1}{x} = \frac{1}{y}$, and hence $\frac{1}{x} \leq \frac{1}{y}$.

Case 2: $y < x$. By assumption (7) (and the hint above) we may multiply both sides by $\frac{1}{y}$ to get $1 < \frac{x}{y}$. Similarly, we may multiply both sides by $\frac{1}{x}$ to get $\frac{1}{x} < \frac{1}{y}$. Hence in this case too, $\frac{1}{x} \leq \frac{1}{y}$. □

(4)(a)[10 pts] Prove, using a proof by contradiction (and use cases within the proof by contradiction), that for all real numbers x and y: if $x < y$, then $x < \frac{x+y}{2}$. Preformalize the statement to be proved along with the negation.

Preformalization: $\forall x \forall y(x < y \to x < \frac{x+y}{2})$; Negation: $\exists x \exists y(x < y \wedge x \geq \frac{x+y}{2})$

Proof. Suppose by way of contradiction that there exist real numbers x and y such that $x < y$ and $x \geq \frac{x+y}{2}$. Then either $x = \frac{x+y}{2}$ or $x > \frac{x+y}{2}$.

Case 1: $x = \frac{x+y}{2}$. Then $2x = x + y$, and so $x = y$, contradicting that $x < y$.

Case 2: $x > \frac{x+y}{2}$. Then $\frac{x+y}{2} < x$. We may multiply both sides of the inequality by 2 via assumption (7) to get $x + y < 2x$. Now add $-x$ to both sides (assumption (6)) to get $y < x$. This contradicts $x < y$, and the proof is complete. □

(4)(b)[5 pts] Prove that for every real number x, there exists a real number y such that $x < y$. Be sure to justify everything in your proof from the notes or the assumption list. Preformalize first.

Preformalization: $\forall x \exists y(x < y)$

Proof. Let x be an arbitrary real number. From assumption (5), $0 < 1$. Now add x to both sides to get $x < x + 1$ (assumption (6)). □

(4)(c)[5 pts] Disprove that for every real number $x > 0$, $\sqrt{x} < x$. No preformalization necessary here.

Proof. Let $x = \frac{1}{4}$. Then $\sqrt{x} = \frac{1}{2} > \frac{1}{4} = x$. Even easier, you can choose $x = 1$. □

(5)(a)[1 pt each] Circle all of the following which are true. No justification needed.

1. If A is any nonempty set and B is any set, then $A \cup B$ is nonempty.
2. If A and B are any nonempty sets, then $A \cap B$ is also nonempty.
3. There is no set S with the property that S has an element x which is also a subset of S.
4. If S is any set, then $\mathcal{P}(S)$, the power set of S, is nonempty.
5. If A and B are any sets, then $A \subseteq B \backslash A$.

Solution. 1. is true, 2. is false, 3. is false, 4. is true, and 5. is false. □

(5)(b)[7 pts] Prove that for all sets A and B: if $A \backslash B = \varnothing$, then $A \subseteq B$. Preformalize first.

Preformalization: $\forall A \forall B (A \backslash B = \varnothing \rightarrow A \subseteq B)$

Proof. Suppose by way of contradiction that there exists sets A and B such that $A \backslash B = \varnothing$, but A is not a subset of B. Then there exists some object x such that $x \in A$ and $x \notin B$. But then $x \in A \backslash B$, contradicting that $A \backslash B = \varnothing$. □

(5)(c)[8 pts] Recall that a set S is **transitive** if for all objects x and y: if $x \in y$ and $y \in S$, then $x \in S$. Prove that for all sets S: if S is transitive, then for every set x: if $x \in S$, then $x \subseteq S$. No preformalization necessary. You may regard 'object' and 'set' as synonymous).

Proof. Let S be an arbitrary set, and assume that S is transitive. Now let x be an arbitrary set, and assume that $x \in S$. Let y be an arbitrary object and assume that $y \in x$. Then $y \in x$ and $x \in S$. Because S is transitive, $y \in S$. Thus shows that $x \subseteq S$. □

Assumptions

1. basic algebra[1]
2. the sum, difference, product, and negatives of integers are integers,
3. every integer is either even or odd,
4. no integer is both even and odd,
5. $1 > 0$,
6. for all real numbers a, b, and c: if $a < b$, then $a + c < b + c$,
7. for all real numbers a, b, and c: if $a < b$ and $c > 0$, then $ac < bc$,
8. for all real numbers a, b, and c: if $a < b$ and $b < c$, then $a < c$, and
9. for all real numbers a, b, and c: exactly one of $a = b$, $a < b$, $b < a$ holds.
10. every rational number can be expressed in reduced form.
11. the product of two nonzero real numbers is nonzero.
12. if x is an integer and $x|1$, then $x = \pm 1$.

[1]When you do basic algebra, you need NOT say "by assumption 1."; you can just do it in the course of a proof.