Math 3410 Assignment 11 Solutions

(0) Give me the embedded words from the 11/7 and 11/9 lectures.

(1)[10 pts] Use the Cauchy Condensation test to prove that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. You may assume that the sequence $(\frac{1}{n})$ is a decreasing sequence with positive terms. Note that I'm beginning the series/sequence with $n = 1$, but this doesn't change any of the results from the notes. The proof is not very long.

Proof. Let $a_n = \frac{1}{n}$ for every positive integer n, and let n be an arbitrary positive integer. Then $2^n a_{2^n} = 2^n \frac{1}{2^n} = 1$. So $(2^n a_{2^n}) = 1$ for every positive integer n. It follows that $(2^n a_{2^n})$ does not converge to 0, so the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges by the nth term test for divergence. □

(2)[10 pts] Let (a_n) be a sequence of real numbers such that $(a_n) \to 0$. Prove that the series $\sum_{n=0}^{\infty} a_n - a_{n+1}$ converges to a_0. Hint: compute $S_0, S_1, S_2, \ldots$ on scratch paper and see if you can find some cancellations which will give you a nice 'closed form' for S_n. Prove your formula for S_n by induction and then solve the problem. You do NOT need any series tests to do this problem.

Proof. First we prove by induction that $S_n = a_0 - a_{n+1}$, where S_n is the nth partial sum of the series $\sum_{n=0}^{\infty} a_n - a_{n+1}$, where we assume that (a_n) is a real sequence converging to 0.

(i) (base case) $S_0 = a_0 - a_1$ by definition, so this case is simple.

(ii) (inductive step) Let $n \in \mathbb{N}$ and suppose that $S_n = a_0 - a_{n+1}$. Then $S_{n+1} = S_n + a_{n+1} - a_{n+2} = a_0 - a_{n+1} + a_{n+1} - a_{n+2} = a_0 - a_{n+2}$.

Thus $\sum_{n=0}^{\infty} a_n - a_{n+1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a_0 - a_{n+1} = \lim_{n\to\infty} a_0 - \lim_{n\to\infty} a_{n+1} = a_0 - 0 = a_0$. Note that we are using the fact that if $(a_n) \to L$, then every tail also converges to L. □

(3)[10 pts] Let $x \in \mathbb{R}$ be arbitrary, and consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Use the ratio test to prove that this series converges (recall that $0! = 1$ and that for every natural number n, $(n+1)! = n!(n+1)$).

Proof. Let $x \in \mathbb{R}$. Then note that we have $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} |\frac{x}{n+1}| = \lim_{n\to\infty} \frac{|x|}{n+1}$. We have seen that $\lim_{n\to\infty} \frac{1}{n+1} = 0$, and so by Limit Law 1, $\lim_{n\to\infty} \frac{|x|}{n+1} = |x| \cdot 0 = 0$. This shows that the series converges absolutely (remember that the ratio test was stated for series with non-negative terms), and thus the series also converges. □

(4)[10 pts] Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Recall that the **composition** of g and f is the function $g \circ f\colon A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for every $a \in A$. Prove that if f and g are one-to-one, then $g \circ f$ is also one-to-one (this is not a long proof).

Proof. Let A, B, f, and g be as stated above. Assume that f and g are one-to-one. Now let $x, y \in A$ be arbitrary, and assume that $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-to-one, $f(x) = f(y)$; since f is one-to-one, $x = y$. □

(5)[10 pts] Suppose that $f\colon A \to B$ and $g\colon B \to C$ are functions. Prove that if f and g are surjective, then $g \circ f$ is also surjective. Again, this isn't a long proof.

Proof. Let A, B, f, and g be as stated. Assume that f and g are surjective. Now let $c \in C$ be arbitrary. Since g is surjective, there is some $b \in B$ such that $g(b) = c$. Since f is surjective, there is some $a \in A$ such that $f(a) = b$. Thus $g(f(a)) = (g \circ f)(a) = c$, and $g \circ f$ is surjective. □

(6)[BONUS] (it's not worth a ton of points, fyi) Note that $.999999\cdots$ (that is, $.\overline{9}$, the repeating decimal) can be written as $.9 + .09 + .009 + .0009 + \cdots$.

(a)[2 pts] Write $.9 + .09 + .009 + .0009 + \cdots$ as a geometric series. In other words, we can express $.9 + .09 + .009 + \cdots$ in the form $\sum_{n=0}^{\infty} cr^n$. What are c and r? You do not need to justify your answer.

Solution. $c = .9$ and $r = \frac{1}{10}$. □

(b)[1 pt] What is the sum of the series from (a)? Is it different from what you expected? (the latter question isn't graded, but I want you to think about it and answer it)

Solution. The sum is 1. You may feel like this isn't correct, as you may expect the sum to be $.\overline{9}$. But $.\overline{9} = 1$ (this is true, and the above is one proof. Note also that $\frac{1}{3} = .\overline{3}$, and multiply both sides by 3). □