COUNTABLY COVERABLE RINGS

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Abstract. Let R be an associative ring. Then R is said to be *coverable* provided R is the union of its proper subrings (which we do *not* require to be unital even if R is so). One verifies easily that R is coverable if and only if R is not generated as a ring by a single element. In case R can be expressed as the union of a finite number of proper subrings, the least such number is called the *covering number* of R. Covering numbers of rings have been studied in a series of recent papers. The purpose of this note is to study rings which can be covered by a countable collection of subrings.

1. Introduction

It is an elementary textbook exercise to show that no group is the union of exactly two proper subgroups. On the other hand, if G is a group which is not cyclic, then it is immediate that $G = \bigcup_{g \in G} \langle g \rangle$ represents G as a union of proper (cyclic) subgroups. There has been considerable attention in the literature paid to groups G which can be expressed as the union of a *finite* number of proper subgroups. When it exists, the least positive integer n for which G can be expressed as the union of n proper subgroups is called the *covering number* of G. One of the main objectives in this area of research is to determine the integers which can be realized as the covering number of some group and, given a class $\mathcal{C}$ of groups, determine the covering numbers of the groups in $\mathcal{C}$. For example, Tomkinson proved in [18] that every non-cyclic solvable group has covering number $p^n + 1$ for some prime p and positive integer n. However, it is not known whether the complement of the set of positive integers which are covering numbers of groups is finite or infinite. For further reading on group coverings, we refer the reader to [2], [7], and [18]; the second reference contains history and many recent developments on the topic.

Given the interest in the covering numbers of groups, it is not surprising that other algebraic structures have also received analogous attention. For example, Khare determined the covering number of a vector space (where we are interested in unions of proper subspaces instead of subgroups, of course) in [10]. In a series of recent papers, the question of the coverability of rings has been explored (see [3], [12], [14], [16], [17], and [19]). In this context, an associative ring with identity is said to be *coverable* if R can be written as the union of its proper subrings (such subrings are *not* assumed to be unital). Analogous to groups, a ring is coverable if and only if it is not generated (as a ring) by a single element. In case a ring R can be so expressed as a finite union, the smallest positive integer n for which R can be expressed as a union of n proper subrings is called the *covering number* of R, and is denoted by $\sigma(R)$. In case R cannot be expressed as the union

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of finitely many proper subrings (which occurs, for example, if R is singly generated), the covering number is infinite, and we write $\sigma(R) = \infty$.

The purpose of this note is to study coverable rings with infinite covering number. The main question we explore in this paper is the following:

Which rings R have the property that R can be expressed as a countable union of proper subrings?

Our primary objective is to introduce the problem and prove some fundamental results. Throughout the paper, a *cover* of a ring R (group G) is a collection of proper subrings of R whose union is R (a collection of proper subgroups of G whose union is G). Observe trivially that every ring which cannot be generated (as a ring) by a single element is coverable: one simply takes the union of all singly generated subrings to obtain a cover. Analogously, a group is coverable if and only if it is not cyclic.

For the purposes of this note, let us call a ring which can be expressed as a finite union of proper subrings *finitely coverable*; such a collection of proper subrings will be called a *finite cover*. Analogously, rings which can be expressed as a countable union of proper subrings will be called *countably coverable*; such a collection of proper subrings will be called a *countable cover*.

We conclude the introduction by setting a few conventions and recalling some fundamental definitions. Throughout, we assume that all rings are associative. We will often focus on unital, commutative rings; when we do so, we shall explicitly mention these additional assumptions. However, unless otherwise specified, **subrings will not be assumed unital even in case the ambient ring is so.** We denote the set of *natural numbers*, *integers*, and *rational numbers* by ω, $\mathbb{Z}$, and $\mathbb{Q}$, respectively, with the convention that $0 \in \omega$.[1] If R is a ring and M is an R-module, then we denote the *injective envelope* of M by $E(M)$. If N is a submodule of M, then we say that N is *essential* in M if N has nontrivial intersection with every nontrivial submodule of M. Finally, recall that for a prime p, the *quasi-cyclic group of type* p^∞ is the direct limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$ as $n \to \infty$.

2. Countably coverable groups and semigroups

As stated in the introduction, it is our purpose to study rings which can be expressed as a countable union of proper subrings. Note that if there is a non-cyclic abelian group G which cannot be expressed as a countable union of proper subgroups, then the zero ring $G(0)$ would be an example of a commutative ring which is not singly generated and which cannot be countably covered. Thus it is natural to ask if every non-cyclic abelian group is countably coverable. We answer this question in the affirmative below. It is likely that this result is in the literature, but we could not locate a source. Therefore, we give a short proof. In fact, we prove a bit more (we will utilize this result later in the paper).

Proposition 1. *Every non-cyclic abelian group G is countably coverable. Moreover, if G is not finitely generated, then G can be covered by a chain $H_0 \subsetneq H_1 \subsetneq \cdots$ of proper subgroups.*

[1] The reason for using the notation ω instead of $\mathbb{N}$ is to utilize the fact that ω (the set of natural numbers) is the first infinite ordinal (cardinal) and that $\omega \leq \kappa$ for every infinite cardinal κ, a result of which we shall make use throughout this note.

Proof. Let G be a non-cyclic abelian group. If G is countable, then G is the union of its cyclic subgroups, and thus is countably coverable. Suppose now that G is countable and not finitely generated. Enumerate G as follows: $G = \{g_0, g_1, g_2, \ldots\}$. Now let H_n be the subgroup of G generated by $g_0, \ldots, g_n$. Because G is not finitely generated, each H_n is a proper subgroup of G. Moreover, we may assume without loss of generality that $H_n \neq H_m$ for $n \neq m$ (by discarding redundancies). Clearly $H_0 \subsetneq H_1 \subsetneq H_2 \cdots$ and the union of the H_i is equal to G.

Next, assume that G has uncountable cardinality κ, and let $E(G)$ be the injective envelope of G. It is well known that G is essential in $E(G)$ and that $E(G)$ is isomorphic to a direct sum of copies of $\mathbb{Q}$ and $C(p^\infty)$ for various prime p (see the classic text [6] for details). Basic cardinal arithmetic yields that $E(G) = \bigoplus_{i<\kappa} G_i$, where each G_i is isomorphic to either $\mathbb{Q}$ or to some quasi-cyclic group. For every $n < \omega$, let $K_n := \{(g_i) \in E(G) \colon g_k = 0 \text{ for } n < k < \omega\}$. Clearly $K_i \subsetneq K_j$ for $i < j < \omega$. Now set $H_n := K_n \cap G$ for $n < \omega$. We claim that $\{H_n \colon n < \omega\}$ is a cover of G by proper subgroups. Viewing G as a subgroup of $E(G)$, we see that every element of G is a sequence $(g_i \colon i < \kappa)$ with finite support such that $g_i \in G_i$ for each i. It is now immediate that every $g \in G$ is a member of some H_n. Next, for every $i < \kappa$, let $\zeta_i \colon G_i \to E(G)$ be the natural embedding. Now let $m < n < \omega$. Then observe that $\zeta_m(G_m) \subseteq K_m$ and $\zeta_n(G_n) \subseteq K_n$, but $\zeta_n(G_n) \cap K_m = \{0\}$. Further, $\zeta_n(G_n) \cap G$ is a nontrivial subgroup of H_n by essentiality of G. However, $\zeta_n(G_n) \cap H_m = \{0\}$. From this, it follows that $H_m \subsetneq H_n$, and it follows that each H_i is proper. The proof is now complete. □

Remark 1. A natural question arises: is every non-cyclic (possibly nonabelian) group countably coverable? The answer, in fact, is no. Saharon Shelah famously constructed a so-called *Jónsson group* G of cardinality ω_1 (the first uncountable cardinal). This is a group of cardinality ω_1 with all proper subgroups countable (see [15]). If G were countably coverable, then G would be a countable union of countable subsets, thus itself countable. In fact, Shelah's group G has the stronger property that every proper subsemigroup is also countable. Hence G cannot be countably covered by proper subsemigroups.

Despite the above obstructions, we have the following corollary, of which we shall make use shortly.

Corollary 1. *The following hold.*

1. *Every non-cyclic commutative monoid is countably coverable by proper subsemigroups.*
2. *Every non-cyclic, cancellative, commutative semigroup S is countably coverable by proper subsemigroups. If S is not finitely generated, then we can choose the cover to be a chain of order type ω as in Proposition 1.*
3. *Every zero ring on a non-cyclic abelian group is countably coverable.*
4. *Let R be a commutative ring with identity such that $(R, +)$ and $(R, \cdot)$ are not cyclic as groups and monoids, respectively. Then both $(R, +)$ and $(R, \cdot)$ are countably coverable as groups and multiplicative monoids, respectively.*

Proof. We verify each claim in succession.

(1) Let M be a non-cyclic (as a semigroup) commutative monoid. If M is countable, clearly M is countably coverable. So assume M is uncountable. Suppose that M is a group. As M is

uncountable, clearly M is not a cyclic group, and we are done by the previous proposition. So assume that M is not a group. Let I be the collection of invertible elements of M and let N be the set of non-invertible elements of M. Then I and N are easily checked to be (proper) subsemigroups of M with $M = I \cup N$. Thus M is countably coverable by subsemigroups (with covering number 2 in this case).

(2) Let S be a non-cyclic countable, cancellative, commutative semigroup. As in the proof of Proposition 1, if S is countable, S is the union of its cyclic subsemigroups, hence countably coverable. If S is countable but not finitely generated, one uses the same argument as the one presented in Proposition 1 to deduce that S can be expressed as the union of a chain (with order type ω) of proper subsemigroups. Now suppose that S is uncountable, and let $\mathcal{Q}(S)$ be the group of differences (viewing S additively) of S. Let $H_0 \subsetneq H_1 \subsetneq \cdots$ be a cover of $\mathcal{Q}(S)$ furnished by Proposition 1, and for each i, set $S_i := H_i \cap S$. If any $S_i = S$, then $S \subseteq H_i$. As H_i is a group, it follows that $\mathcal{Q}(S) \subseteq H_i$, and hence $H_i = \mathcal{Q}(S)$, a contradiction. Therefore, (discarding any redundancies) $S_0 \subsetneq S_1 \subsetneq \cdots$ is a cover of S.

(3,4) Obvious. □

3. Countably coverable rings

3.1. **Fundamental results.** We open this section with several examples. Recall that a ring is countably coverable if it can be expressed as a countable union of proper subrings (which are not assumed unital even if the ambient ring is so).

Example 1. *Let R be a ring of characteristic 0 such that the additive group $(R, +)$ is torsion. Then R is countably coverable.*

Proof. Let R be as described. For every positive integer n, let $R_n := \{r \in R : nr = 0\}$. One checks at once that each R_n is a subring of R. Since R has characteristic 0, each R_n is a proper subring of R. Finally, as $(R, +)$ is torsion, it follows that $R = \bigcup_{n \in \mathbb{Z}^+} R_n$, showing that R is countably coverable. □

Example 2. *Let R be an uncountable ring such that $|R|$ has countable cofinality. Then R is countably coverable.*

Proof. Choose a strictly increasing sequence $(\kappa_n : n \in \mathbb{Z}^+)$ of infinite cardinals which are cofinal in $|R|$. Now choose, for every positive integer n, a subset X_n of R of cardinality κ_n such that the union of the X_n is all of R. For each n, let R_n be the subring of R generated by X_n. Then each R_n has cardinality $\kappa_n < |R|$, thus is proper. It is clear that the union of the R_n is R. □

Example 3. *Let G be Shelah's Jónsson group of size ω_1 mentioned in Remark 1. Then the group ring $\mathbb{Z}[G]$ is not countably coverable.*

Proof. The elements of $\mathbb{Z}[G]$ are finitely nonzero formal sums $\sum_i a_i g_i$ with $a_i \in \mathbb{Z}, g_i \in G$, from which it follows that G generates $\mathbb{Z}[G]$ as a ring. Thus no proper subring of $\mathbb{Z}[G]$ can contain all of G. Suppose by way of contradiction that $\mathbb{Z}[G]$ is covered by countably many proper subrings

$R_1, R_2, \ldots$. For each $n \in \mathbb{Z}^+$, let $S_n := G \cap R_n$. Then each nonempty S_n is a proper subsemigroup of G, because if $G = S_n$, then R_n contains all of G and hence $R_n = \mathbb{Z}[G]$. Now, each element of G lies in some S_n. So, G is equal to the union of all the S_n. But then G is the union of countably many proper subsemigroups, hence is countable by Remark 1. □

Remark 2. In light of Example 3, it is natural to enquire about possible commutative analogs of the above construction. It is well known that there are no uncountable abelian Jónsson groups (groups whose proper subgroups are of strictly smaller cardinality than the group; thus Shelah's group G above is non-abelian), and here is a short argument: let G be an uncountable abelian group and let $E(G) := \bigoplus_{i<|G|} G_i$ be the injective envelope of G as in the proof of Proposition 1. As G is essential in $E(G)$, we see that $(\bigoplus_{0<i<|G|} G_i) \cap G$ is a proper subgroup of G of size $|G|$. Despite this fact, if there is a commutative ring R of cardinality ω_1 with all proper subrings countable, then it follows that R is not countably coverable. But no such ring exists. Indeed, Robert Gilmer and Bill Heinzer have shown in [8] that every uncountable commutative ring (commutative ring with identity) has a proper uncountable subring (unital subring) of the same cardinality. Moreover, it is not known if there exists a noncommutative ring of size ω_1 with all proper subrings countable; see [4] and [11] for further details.

A natural question, given the above example and remarks, is the following: is every non-singly generated commutative ring countably coverable? Though we do not know the answer, we show that "most" commutative rings with identity are countably coverable. We begin with an easy but useful lemma which appears in some form throughout the cited literature, and present the short proof.

Lemma 1. *Let S be a ring, I a two-sided ideal of S, and $\varphi\colon S \to S/I$ the canonical epimorphism. If $\mathcal{C}$ is a cover of S/I, then $\varphi^{-1}(\mathcal{C}) := \{\varphi^{-1}(R)\colon R \in \mathcal{C}\}$ is a cover of S.*

Proof. Let $R \in \mathcal{C}$ be arbitrary. Since $\varphi(0) = 0 \in R$, $0 \in \varphi^{-1}(R)$. Further, if $x, y \in S$ are such that $\varphi(x), \varphi(y) \in R$, then $\varphi(x + y) = \varphi(x) + \varphi(y) \in R$. Similarly, $\varphi(xy) = \varphi(x)\varphi(y) \in R$. This shows that $\varphi^{-1}(R)$ is a subring of S for every $R \in \mathcal{C}$. If there is some $R \in \mathcal{C}$ for which $\varphi^{-1}(R) = S$, then $\varphi(s) \in R$ for every $s \in S$. As φ is onto, this implies that $R = S/I$, contradicting that R is proper. Finally, let $s \in S$ be arbitrary. Then $\varphi(s) \in S/I$, so $\varphi(s) \in R$ for some $R \in \mathcal{C}$. It follows that $s \in \varphi^{-1}(R)$, and this proves that $\varphi^{-1}(\mathcal{C})$ is a cover of S. □

Next, we state another result from the literature.

Lemma 2. [16, Proposition 3.1] *Suppose that R is a unital ring and that $\sigma(R) < \infty$ (that is, R can be covered by finitely many proper subrings). Then there exists a proper two-sided ideal I of R such that R/I is finite and $\sigma(R) = \sigma(R/I)$.*

The following interesting consequence follows immediately from the lemma: no field is finitely coverable. To see this, recall that every finite field has a cyclic multiplicative group, hence is singly generated as a ring. Moreover, infinite fields do not possess nontrivial, finite residue rings. Hence by Lemma 2, they are not finitely coverable. In fact, we can prove a bit more. We shall require the following lemma.

Lemma 3. [13, Corollary 4] *Let R be an infinite unital ring which satisfies the ascending chain condition on two-sided ideals. Then there exists a prime ideal P of R such that $|R/P| = |R|$.*

Proposition 2. *Let R be an infinite commutative Artinian local ring with identity. Then R is not finitely coverable.*

Proof. Suppose by way of contradiction that there is an infinite commutative Artinian local ring R with identity that is finitely coverable. Then by Lemma 2, there is a proper ideal I of R such that R/I is finite. Since R is a local Artinian ring, R possesses a unique prime ideal P, and $I \subseteq P$. By Lemma 3 above, $|R/P| = |R|$. But $I \subseteq P$ implies that $|R/P| \leq |R/I| < \omega$, and this is a contradiction to Lemma 3. □

On the other hand, all infinite commutative Artinian rings with identity are *countably* coverable, which we obtain as a corollary to our first theorem below.[2] First, we pause to present a proposition on sums of units using the notion of coverability.

Proposition 3. *Let R be a commutative ring with identity.*

(1) *If R has but countably many maximal ideals and not every $r \in R$ is a sum of units, then R is countably coverable.*
(2) *If R is semilocal with all residue fields infinite, then every $r \in R$ is a sum of units.*
(3) *If R is a Noetherian semilocal domain of cardinality greater than 2^ω, then every element of R is a sum of units.*

Proof. Let R be a commutative ring with identity.

(1) Suppose R has but countably many maximal ideals and that not every element of R is a sum of units. Now let $U := \{u_1 + u_2 + \cdots + u_n : n \in \mathbb{Z}^+, u_i \in R^\times\}$. Then one checks easily that U is a subring of R, thus proper by assumption. It follows that R is covered by U and all maximal ideals of R.

(2) Let R be semilocal with all residue fields infinite. Suppose by way of contradiction that not every element of R is a sum of units. Then R is finitely coverable (analogous to the argument given in (1)). Lemma 2 implies that there is a proper ideal I of R such that R/I is finite. Now, $I \subseteq M$ for some maximal ideal M of R. Thus $|R/M| \leq |R/I| < \omega$, a contradiction.

(3) Suppose that R is a Noetherian semilocal domain of size greater than 2^ω, and let M be a maximal ideal of R. By [9, Lemma 2.1], we have $|R/M| + \omega \leq |R| \leq |R/M|^\omega$. If R/M is finite, then it follows that R has size at most 2^ω, a contradiction. Thus all residue fields are infinite, and we invoke (2). □

Remark 3. We cannot drop the assumption in Proposition 3(2) about the cardinality of the residue fields completely. For example, $\mathbb{F}_2 \times \mathbb{F}_2$ is a semilocal commutative ring with identity, but $(1, 0)$ is not a sum of units.

[2] We are grateful to Be'eri Greenfeld for a proof that $\mathbb{R}$ is countably coverable. We have extrapolated his argument to obtain Theorem 1.

We now present the first theorem of the paper.

Theorem 1. *Let D be a commutative domain (not assumed to have an identity). Then D is countably coverable if and only if D is not singly generated. Moreover, if D is not finitely generated, then D can be covered by a chain of proper subrings of order type ω.*

Proof. Let D be a commutative domain. It is clear that if D is countably coverable, then D is not singly generated. Let us suppose now that D is not singly generated. If D is countable, then D is countably covered by all singly-generated subrings. Now suppose that D is countable and not finitely generated (as a ring); let $\{d_0, d_1, d_2, \ldots\}$ be an enumeration of D, and let R_n be the subring of D generated by $d_0, \ldots, d_n$. Because D is not finitely generated, each R_n is a proper subring of D. By discarding redundancing, we obtain a cover of D by a a chain of subrings order-isomorphic to ω.

Suppose now that D is uncountable, and let $\mathcal{Q}(D)$ be the fraction field of D, with algebraic closure $\mathcal{Q}(D)^a$. Then up to isomorphism, D is a subdomain of $\mathcal{Q}(D)^a$. Let F be the field generated by $1 \in \mathcal{Q}(D)$. Now let β be a maximal subset of D which is algebraically independent over F (which exists by Zorn's Lemma). Basic cardinal arithmetic along with the fact that a polynomial over a domain has but finitely many roots yields that $|\beta| = |D| := \kappa$. Enumerate β relative to κ by setting $\beta := \{b_i : i < \kappa\}$. For every $n \in \mathbb{Z}^+$, let $K_n := F(\beta \backslash \{b_k : n < k < \omega\})^a \subseteq \mathcal{Q}(D)^a$, and set $H_n := K_n \cap D$. Note that K_n is a chain of subrings of $\mathcal{Q}(D)^a$ of order type ω. We claim that $\{H_n : n \in \mathbb{Z}^+\}$ is a cover of D by a chain of proper subrings of order type ω. We first show that each H_n is a proper subring of D. Indeed, suppose $H_n = D$ for some positive integer n. Then $D \subseteq F(\beta \backslash \{b_k : n < k < \omega\})^a$, and so b_{n+1} is algebraic over $F(\beta \backslash \{b_k : n < k < \omega\})$, contradicting that β is algebraically independent over F. Now let $d \in D$ be arbitrary. The maximality of β implies that d is algebraic over $F(\beta)$. But then clearly d is algebraic over $F(\beta')$ for some finite $\beta' \subseteq \beta$. Now simply choose $n \in \omega$ such that $\beta' \subseteq \{b_0, b_1, \ldots, b_n\} \cup \{b_i : \omega \leq i\}$. It is clear that $d \in K_n$, and so $d \in H_n$. Finally, the algebraic independence of β immediately yields that $H_i \subsetneq H_j$ for $0 \leq i < j < \omega$, and the proof is complete. □

Remark 4. An analogous question to the one in the last subsection arises: is this theorem true for possibly noncommutative prime rings? The answer is no. Indeed, Connell proved in [5, p. 675] (and this is well known) that if R is a unital ring and G is a group, then the group ring $R[G]$ is prime if and only if R is prime and G has no nontrivial finite normal subgroups. Yet another remarkable property of the Jónsson group G of size ω_1 constructed by Shelah is that G is simple. Thus $\mathbb{Z}[G]$ is a non-singly generated prime ring which is not countably coverable.

We pause to present the following corollary of Theorem 1.

Corollary 2. *Let R be a commutative ring with identity.*

(1) *If R is Noetherian and not singly generated, then R is countably coverable.*
(2) *If R is an infinite field, then R can be countably covered by a chain of proper subrings of order type ω.*
(3) *If the additive group of R is divisible, then R is countably coverable.*

Proof. Let R be a commutative ring with identity.

(1) Suppose that R is Noetherian and not singly generated. If R is countable, we are clearly done. Thus suppose that R is uncountable, and let P be a prime ideal of R such that $|R/P| = |R|$ furnished by Lemma 3. As R/P is uncountable, clearly R/P is not singly generated, so by Theorem 1, R/P is countably coverable. Lemma 1 finishes the argument.

(2) Now assume that R is an infinite field. It is well-known that R is not finitely generated as a ring (this is a standard result of commutative algebra; see [1] for details). Apply Theorem 1.

(3) Assume now that the additive group of R is divisible, that is, $nR = R$ for every positive integer n. It is well-known that every (nontrivial) divisible abelian group is infinite (this is a textbook exercise) and easy to see that divisibility is preserved by factor groups. So let M be a maximal ideal of R. Then by the above comments, R/M is an infinite field. By (2), R/M is countably coverable. Finally, apply Lemma 1. □

Before proceeding, we shall require a lemma characterizing the singly generated commutative domains, which are obstructions to countable coverability.

Lemma 4. *Let D be a commutative integral domain (not assumed to have an identity). Then D is singly generated if and only if one of the following holds:*

(1) *D is a finite field.*
(2) *$D \cong X\mathbb{F}_p[X] := \{Xf(X) \colon f(X) \in \mathbb{F}_p[X]\}$ ($\mathbb{F}_p$ is the field with p elements, p a prime).*
(3) *$D \cong \alpha\mathbb{Z}[\alpha] := \{\alpha f(\alpha) \colon f(X) \in \mathbb{Z}[X]\}$ for some $\alpha \in \mathbb{C}$.*

Proof. To begin, recall that every finite field has a cyclic multiplicative group, and thus every finite field is singly generated. It is immediate that the rings in (2) and (3) are generated by X and α, respectively.

Now suppose that D is a commutative, singly generated domain. We first consider the case where D has characteristic p, p a prime. Let $d \in D$ generate D, and let $\varphi \colon X\mathbb{F}_p[X] \to D$ be defined by $\varphi(Xf(X)) := df(d)$. Then φ is a surjective ring homomorphism, and hence $X\mathbb{F}_p[X]/K \cong D$, where K is the kernel of the map. If the kernel is trivial, then $D \cong X\mathbb{F}_p[X]$ and we are done, so assume that K is nontrivial. Now, K is also an ideal of $\mathbb{F}_p[X]$, and since K is nonzero, $\mathbb{F}_p[X]/K$ is finite. It follows that $X\mathbb{F}_p[X]/K$ is also finite, and so D is a finite field. Now suppose that D has characteristic 0 and is generated by α as a ring. Since D is a countable domain of characteristic 0, up to isomorphism, D is a subring of $\mathbb{C}$. To see this, simply adjoin 2^ω many variables to D and take the quotient field, then the algebraic closure. Then we obtain an algebraically closed field F of characteristic 0 of transcendence degree 2^ω. By Steinitz' theorem (any two algebraically closed fields of the same transcendence degree and the same characteristic are isomorphic), we see that $F \cong \mathbb{C}$, and the result follows. □

As we stated earlier, we do not know if there exist any non-singly generated commutative rings R with identity which are not countably coverable. However, our next proposition collects some necessary conditions for R to be so.

Proposition 4. *Let R be a commutative ring with identity that is not singly generated. If R is not countably coverable, then the following hold:*

(1) *R is not Noetherian.*
(2) *R/P is singly generated for every prime ideal P of R.*
(3) *All residue fields of R are finite.*
(4) *If R has positive characteristic, then R has Krull dimension zero.*
(5) *If R has characteristic zero, then R has Krull dimension one.*

Proof. Let R be as stated.

(1) This is immediate from Corollary 2(1).

(2) From Lemma 1 and Theorem 1.

(3) By (2), all residue fields of R are singly generated, and so by Lemma 1 and Corollary 2(2), all residue fields are finite.

(4) Suppose that R has positive characteristic, and let P be a prime ideal of R. Then R/P is a singly generated unital domain of positive characteristic. Thus by Lemma 4, R/P is a field. It follows that P is maximal.

(5) Finally, assume that R has characteristic zero. Note that $\mathbb{Z}\backslash\{0\}$ is a multiplicatively closed subset of R (up to isomorphism), and so there exists a prime ideal P of R such that $P \cap \mathbb{Z} = \{0\}$. It follows that R/P also has characteristic 0, thus is infinite; by (2), R/P is singly generated. Lemma 1 and Corollary 2 imply that R/P is not a field, thus P is not maximal. This shows that R has Krull dimension at least one. Suppose by way of contradiction that there exist prime ideals $P_1 \subsetneq P_2 \subsetneq P_3$ of R. Then $D := R/P_1$ is singly generated, unital, and of Krull dimension at least two. Lemma 4 implies that $D \cong \alpha\mathbb{Z}[\alpha]$ for some $\alpha \in \mathbb{C}$ (since the rings in (1) and (2) of Lemma 4 have Krull dimension zero and are non-unital, respectively). But because D is unital, it follows that $D \cong \mathbb{Z}[\alpha]$. Let $\varphi \colon \mathbb{Z}[X] \to D$ be a surjective homomorphism with kernel K. It is easy to see that $\mathbb{Z}[X]$ is not singly generated as a ring, and so since $\mathbb{Z}[X]/K \cong D$, we deduce that K is nontrivial. However, this implies that $\mathbb{Z}[X]/K$ has Krull dimension at most one (since $\mathbb{Z}[X]$ has Krull dimension two), contradicting our assumption above that D has Krull dimension at least two. □

3.2. **Behavior relative to ring constructions.** The purpose of the final part of the paper is to catalog some ring constructions which both preserve and (have the potential to) destroy countable coverability.

Proposition 5. *Let R be a ring.*

(1) *If R is countably coverable and S is any ring, then $R \times S$ is also countably coverable.*
(2) *If R does not have an identity, yet R is countably coverable, then the ring R^1 obtained by adjoining an identity to R is also countably coverable.*
(3) *If R is a commutative ring with identity and $n > 1$ is an integer, then the full matrix ring $M_n(R)$ is countably coverable.*

Proof. Let R be a ring.

(1) Suppose that R is countably covered by $\mathcal{C}$ and let S be any ring. Then it is immediate that $\{T \times S \colon T \in \mathcal{C}\}$ is a countable cover of $R \times S$.

(2) Assume that R does not have an identity but is covered by a countable collection $\mathcal{C}$ of proper subrings. Note that R is naturally a module over $\mathbb{Z}/n\mathbb{Z}$, where n is the characteristic of R (note that $n = 0$ is possible). Set $S := \mathbb{Z}/n\mathbb{Z}$, and consider the so-called Dorroh extension R^1 given by $S(+)R$. We remind the reader that the ground set of R^1 is $S \times R$, with addition and multiplication defined by $(s_1, r_1) + (s_2, r_2) := (s_1 + s_2, r_1 + r_2)$ and $(s_1, r_1) \cdot (s_2, r_2) := (s_1 s_2, r_1 s_2 + s_1 r_2 + r_1 r_2)$. Note that R embeds into R^1 via the map $r \mapsto (0, r)$ and that we have $(1, 0) \cdot (s, r) = (s, 0 \cdot s + 1 \cdot r + 0 \cdot r) = (s, r)$ for any $(s, r) \in R^1$. One checks at once that $\mathcal{C}' := \{S \times T \colon T \in \mathcal{C}\}$ is a countable cover of R^1 by proper subrings.

(3) Now assume that R is a commutative unital ring and let $n > 1$ be an integer. Further, let J be a maximal ideal of R. Note that by the fundamental theorem of ring homomorphisms, we have $M_n(R)/M_n(J) \cong M_n(R/J)$. It now suffices by Lemma 1 to show that $M_n(R/J)$ is countably coverable. Set $F := R/J$. If F is finite, then since $M_n(F)$ is noncommutative, it follows that $M_n(F)$ is not singly generated, thus is finitely coverable. Now suppose that F is infinite. Corollary 2(2) implies that F is covered by a countably infinite chain $\mathcal{C}$ of proper subrings. From this fact, it follows that every member of $M_n(F)$ is a member of $M_n(T)$ for some $T \in \mathcal{C}$. Hence $\{M_n(T) \colon T \in \mathcal{C}\}$ is a cover of $M_n(F)$, concluding the proof. □

Our next proposition presents a negative result.

Proposition 6. *Countable coverability is not, in general, preserved by factor rings or localizations.*

Proof. Simply take a domain D which is not singly generated but which has a finite residue field. Recalling that every finite field has a cyclic multiplicative group (hence is singly generated as a ring), this shows that factor rings do not in general preserve countable coverability. As for localizations, simply consider a finite, reduced commutative ring R with identity which is not singly generated (such as $\mathbb{F}_2 \times \mathbb{F}_2$). Then R is finitely coverable. Next, let P be a minimal prime ideal of R. Then as is well-known, the localization R_P is a field (see [1] for details) and is also finite, hence is not countably coverable. □

We conclude the paper with a positive result, and close with two open questions. First, a lemma.

Lemma 5. *Let D be a unital integral domain, and suppose that S is a nontrivial subsemigroup of $\mathbb{Z}$. Then the semigroup ring $D[S]$ is countably coverable.*

Proof. Let D and S be as stated. Suppose first that S is a group. Then $S \cong \mathbb{Z}$, so we show first that $D[\mathbb{Z}] \cong D[X, X^{-1}]$ is countably coverable. By Theorem 1, it suffices to prove that $D[X, X^{-1}]$ is not singly generated. For suppose that f is a generator. Then $m_1 f + m_2 f^2 + \cdots + m_k f^k = 1$ for some integers $m_1, \ldots, m_k$ and positive integer k. But then f is a unit, and so $f = X^b$ for some integer b. However, it is clear that X^b cannot generate $D[X, X^{-1}]$ as a ring (the only powers of X in the ring generated by X^b are the powers X^{nb} where n is a positive integer), and so this concludes the first part of the proof.

Now suppose that S is not a group. Then as is well-known, either $S \subseteq \omega$ or $S \subseteq -\omega$. Via the isomorphism $s \mapsto -s$, we may assume without loss of generality that $S \subseteq \omega$, and hence $D[S]$ is a subdomain of $D[X]$. As above, suppose that f is a generator of $D[S]$. Then f is a unit of $D[S]$, hence of $D[X]$, and it follows that f is a unit of D. But this implies that $D[S] = D$, contradicting that S is nontrivial. □

Proposition 7. *Let R be a commutative ring with identity.*

(1) *The power series ring $R[[X]]$ is countably coverable.*
(2) *If S is an infinite commutative, cancellative semigroup, then the group ring $R[S]$ is countably coverable (and thus, the polynomial ring $R[X]$ is countably coverable).*

Proof. Let R be a commutative ring with identity.

(1) Let M be a maximal ideal of R, and note that $R[[X]]/M[[X]] \cong (R/M)[[X]]$. By Lemma 1, it suffices to prove that $(R/M)[[X]]$ is countably coverable. Toward this end, simply observe that $(R/M)[[X]]$ is an uncountable domain, thus is not singly generated as a ring. Now apply Theorem 1.

(2) Now let S be an infinite commutative, cancellative semigroup, and consider the group ring $R[S]$. Suppose first that S is not finitely generated. Then by Corollary 1(2), there exists a countable cover $\mathcal{C}$ of S which is a chain. It follows that $\{R[T] \colon T \in \mathcal{C}\}$ is a countable cover of $R[S]$.[3] Next, let us assume that S (denoted additively) is finitely generated, and let $\mathcal{Q}(S)$ be the group of differences of S. Then $\mathcal{Q}(S)$ is also finitely generated; because $\mathcal{Q}(S)$ is infinite, the fundamental theorem of finitely generated abelian groups yields that $\mathcal{Q}(S) \cong A \oplus H$ for some nontrivial free abelian group A and finite abelian group H. Let $\pi \colon A \oplus H \to \mathbb{Z}$ be a surjective projection. Now let P be a prime ideal of R, and observe that we obtain surjective homomorphisms $R[S] \to (R/P)[S] \to (R/P)[T]$, where T is the image of S (viewed canonically as a subsemigroup of $\mathcal{Q}(S)$) under π. To conclude the proof, apply Lemma 1 and Lemma 5. □

Remark 5. Observe that if S is finite, $R[S]$ need not be countably coverable. Indeed, let F be a finite field and let G be a finite cyclic group whose cardinality is relatively prime to $|F| - 1$. It is easy to see that $F[G]$ is singly generated, and thus not countably coverable. Moreover, Example 3 shows that we cannot drop the assumption that S is commutative entirely.

3.3. **Open questions.** We close with the following two open questions.

Question 1. Is every non-singly generated commutative ring with identity countably coverable?

More generally,

Question 2. Can the countably coverable rings be classified?

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[3]In this case, R need not be commutative nor be unital.

References

[1] M. Atiyah, I.G. Macdonald, *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.
[2] M. Bhargava, *Groups as unions of proper subgroups*, Amer. Math. Monthly **115** (2009), no. 5, 413–422.
[3] M. Cai, N. Werner, *Covering numbers of upper triangular matrix rings over finite fields*, Involve **12** (2019), no. 6, 1005–1013.
[4] E. Coleman, *Jónsson groups, rings and algebras*, Irish Math. Soc. Bull. **36** (1996), 34–45.
[5] G. Connell, *On the group ring*, Canad. J. Math. **15** (1963), 650–685.
[6] L. Fuchs, *Abelian groups.* International Series of Monographs on Pure and Applied Mathematics Pergamon Press, New York-Oxford-London-Paris 1960.
[7] M. Garonzi, L-C. Kappe, E. Swartz, *On integers that are covering numbers of groups*, Experimental Mathematics, `https://doi.org/10.1080/10586458.2019.163426`, 2019.
[8] R. Gilmer, W. Heinzer, *On the cardinality of subrings of a commutative ring*, Canad. Math. Bull. **29** (1986), no. 1, 102–108.
[9] K. Kearnes, G. Oman, *Cardinalities of residue fields of Noetherian integral domains*, Comm. Algebra **38** (2010), no. 10, 3580–3588.
[10] A. Khare, *Vector spaces as unions of proper subspaces*, Linear Algebra Appl. **431** (2009), no. 9, 1681–1686.
[11] A. Kostinsky, *Some problems for rings and lattices within the domain of general algebra.* Thesis (Ph.D.), University of California, Berkeley, 1969.
[12] A. Lucchini, A. Maróti, *Rings as the unions of proper subrings*, Algebr. Represent. Theory **15** (2012), no. 6, 1035–1047.
[13] G. Oman *Small and large ideals of an associative ring*, J. Algebra Appl. **13** (2014), no. 5, 1350151, 20 pp.
[14] G. Peruginelli, N. Werner, *Maximal subrings and covering numbers of finite semisimple rings*, Comm. Algebra **46** (2018), no. 11, 4724–4738.
[15] S. Shelah, *On a problem of Kurosh, Jónsson groups, and applications*, Word Problems II (Proc. Conf. Oxford, 1976), North-Holland (Amsterdam, 1980), 373–394.
[16] E. Swartz, N. Werner, *Covering numbers of commutative rings*, J. Pure. Appl. Algebra **225** (2021), no. 8, 106622.
[17] E. Swartz, N. Werner, *The covering numbers of rings*, `https://arxiv.org/pdf/2112.01667.pdf`, 2022.
[18] M. Tomkinson, *Groups as the union of proper subgroups*, Math. Scand. **81** (1997), no. 2, 191–198.
[19] N. Werner, *Covering numbers of finite rings*, Amer. Math. Monthly **122** (2015), no. 6, 552–566.

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