Math 3410 Lecture 22

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Limits of Functions

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $c \in \mathbb{R}$. Then we say that **the limit as** x **approaches** c **of** $f(x)$ **is** L, and write $\lim_{x \to c} f(x) = L$, provided that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every real number x: if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

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What the limit definition is saying, more informally, is the following: for any $\epsilon > 0$, there is a $\delta > 0$ such that if $x \neq c$ is a real number within a distance of δ of c, then $f(x)$ is within a distance of ϵ of L.

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Let L be an arbitrary real number, and consider the constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = L$ for every real number x.

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Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for every real number x and let $c \in \mathbb{R}$ be arbitrary. Now let $\epsilon > 0$ be a real number. Can you see what we can choose δ to be here? Let $\delta = \epsilon$. Now let $x \in \mathbb{R}$ and suppose that $0 < |x - c| < \epsilon$. We must show that $|f(x) - c| < \epsilon$, that is, $|x - c| < \epsilon$, which is clearly true, and the proof is complete. $\square$

Example (Function Limit Law 2)

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Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $c, L \in \mathbb{R}$. Suppose that $\lim_{x \to c} f(x) = L$. Now let (x_n) be a sequence such that $x_n \neq c$ for any natural number n and such that $(x_n) \to c$.

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Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and suppose that $\lim_{x\to c} f(x) = L$. Now let (x_n) be a sequence such that $x_n \neq c$ for all $n \in \mathbb{N}$ and such that $(x_n) \to c$. We will show that $(f(x_n)) \to L$. So let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to c} f(x) = L$, there is a real number $\delta > 0$ such that (*) for every real number x: if
$0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

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Proof.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and suppose that $\lim_{x\to c} f(x) = L$. Now let (x_n) be a sequence such that $x_n \neq c$ for all $n \in \mathbb{N}$ and such that $(x_n) \to c$. We will show that $(f(x_n)) \to L$. So let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to c} f(x) = L$, there is a real number $\delta > 0$ such that (*) for every real number x: if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Now, because $(x_n) \to c$ and $x_n \neq c$ for every natural number n, it follows that there is some $k \in \mathbb{N}$ such that for every natural number $n \geq k$, we have $0 < |x_n - c| < \delta$.

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Theorem (Sequences to Functions Theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $c, L \in \mathbb{R}$. Suppose that for EVERY sequence (x_n) with the property that $x_n \neq c$ for every n AND for which $(x_n) \to c$, we have $(f(x_n)) \to L$.

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Proof 1.

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Let f, g, c, L, M be as stated above. We will give a direct $\epsilon - \delta$ proof first. Let $\epsilon > 0$.

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Proof 1.

Let f, g, c, L, M be as stated above. We will give a direct $\epsilon - \delta$ proof first. Let $\epsilon > 0$. There exists a real number $\delta_1 > 0$ such that for every real number x: if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$.

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Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are functions, and let $c \in \mathbb{R}$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c}(f+g)(x) = L + M$.

Proof 1.

Let f, g, c, L, M be as stated above. We will give a direct $\epsilon - \delta$ proof first. Let $\epsilon > 0$. There exists a real number $\delta_1 > 0$ such that for every real number x: if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$. Similarly, there exists a real number $\delta_2 > 0$ such that for all real numbers x: if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$. Now let $\delta = \min(\delta_1, \delta_2)$, and suppose that $x \in \mathbb{R}$ and $0 < |x - c| < \delta$. Then note that both $0 < |x - c| < \delta_1$ and $0 < |x - c| < \delta_2$. Thus $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$.

Limits of Functions

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Limits of Functions

Example

Give a direct proof that $\lim_{x \to 1} x^2 = 1$.

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