Math 2150 Assignment 12 Solutions

Throughout, use ONLY the assumptions given in the online notes and/or examples given in the online notes (which you need not reprove) unless specified otherwise. No preformalizations necessary.

(0) Give me the embedded words from the 11/23 lecture.

(1)[10 pts] Prove by induction that for every positive integer n, we have $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Let S be the set of all positive integers n satisfying the above equation.

(i) base case: $1 \in S$: we simply must check that $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$, which is clearly true.

(ii) inductive step: let n be a positive integer, and assume that $n \in S$. Then by definition of S, we have (*) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. We must prove that $n+1 \in S$, that is, we must show that (**) $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$. Toward this end, we begin by adding $(n+1)^2$ to both sides of (*) to get the following: $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} = \frac{(n+1)(n(2n+1)+6(n+1))}{6} = \frac{(n+1)(2n^2+n+6n+6)}{6} = \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$. We have proven (**), and so $n+1 \in S$, as required. $\square$

(2)[10 pts] Prove by induction that for every positive integer n, we have $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Proof. Let S be the set of all positive integers n for which $2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

(i) base case: $1 \in S$: we simply must check that $2^1 = 2^{1+1} - 2$, which is clear.

(ii) inductive step: let n be a positive integer, and assume that $n \in S$. Then by definition of S, we see that (*)$2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$. We must prove that $n+1 \in S$, that is, we must show that (**) $2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} = 2^{n+2} - 2$. Begin by adding 2^{n+1} to both sides of (*) to get $2^1 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1} = 2^{n+1} - 2 + 2^{n+1} = 2^{n+1} + 2^{n+1} - 2 = 2 \cdot 2^{n+1} - 2 = 2^{n+2} - 2$, as desired. $\square$

(3)(a)[3 pts] Find a formula for the expression $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}$ (do this on scratch paper; you do NOT need to show your work here). What I'm looking for here is something like, "$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{3}{n^2+1}$" (of course, that's not correct, but you hopefully get the idea). Start by trying to find a pattern by plugging in small positive integer values for n (but you don't need to show me this; no work required).

Solution $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for every positive integer n. □

(3)(b)[7 pts] Prove that the formula you came up with in (a) is correct by using mathematical induction.

Proof. Let S be the set of all positive integers n for which the above equation holds.

(i) base case: $1\in S$: we must simply check that $\frac{1}{1\cdot 2}=\frac{1}{1+1}$, which is clearly true.

(ii) inductive step: let n be a positive integer, and assume that $n\in S$. Then (*) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$. We must show that (**) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$. Begin by adding $\frac{1}{(n+1)(n+2)}$ to both sides of (*) to produce $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)}=\frac{n(n+2)+1}{(n+1)(n+2)}=\frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2}$, as desired. □