Math 3410 Assignment 14 Solutions

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x + 2$. What is $f^{-1}([2,3])$?

Solution. $f^{-1}([2,3]) = \{r \in \mathbb{R}\colon 2 \leq 3r + 2 \leq 3\} = \{r \in \mathbb{R}\colon 0 \leq 3r \leq 1\} = \{r \in \mathbb{R}\colon 0 \leq r \leq \frac{1}{3}\} = [0, \frac{1}{3}]$. □

(2) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Suppose that A and B are subsets of $\mathbb{R}$ such that $A \cap B = \varnothing$. Prove that $f^{-1}(A) \cap f^{-1}(B) = \varnothing$ as well.

Proof. Let f, A, and B be as stated. Assume that $A \cap B = \varnothing$. Now suppose by way of contradiction that there is some $x \in f^{-1}(A) \cap f^{-1}(B)$. Then $f(x) \in A$ and $f(x) \in B$, that is, $f(x) \in A \cap B$. But this contradicts that $A \cap B = \varnothing$. □

(3) Suppose that S is a nonempty, bounded subset of $\mathbb{R}$. Suppose also that for any $x, y \in S$ with $x < y$, if z is any real number such that $x < z < y$, then $z \in S$. Suppose even further that $\inf(S) \in S$ and $\sup(S) \in S$. Prove that $S = [a, b]$ for some real numbers a and b (hint: what do you think a and b are?). Note that you are proving a set equality, so you have to show that every memeber of S is in $[a, b]$ and that every memeber of $[a, b]$ is in S.

Proof. Let S satisfy the above hypotheses. We will show that $S = [a, b]$, where $a = \inf(S)$ and $b = \sup(S)$. Indeed, let $x \in S$ be arbitrary. Then $\inf(S) \leq x$ since $\inf(S)$ is a lower bound of S. Similarly, since $\sup(S)$ is an upper bound of S, we have $x \leq \sup(S)$. So we have shown that $x \in [a, b]$. Conversely, suppose that $x \in [a, b]$. Then $\inf(S) \leq x \leq \sup(S)$. If $x = \inf(S)$ or $x = \sup(S)$, then $x \in S$ by our assumption that both the inf and sup of S are members of S. So we now may assume that $\inf(S) < x < \sup(S)$. Then by the condition on S, we see that $x \in S$, and the proof is complete. □

(4) Prove that $\mathbb{R}$ cannot be written as the union of two nonempty, disjoint open sets (remember that being disjoing means the sets have no element in common). Hint: suppose that $\mathbb{R} = A \cup B$, where A and B are nonempty, disjoint open subset of $\mathbb{R}$. Without loss of generality, there exists $a \in A$ and $b \in B$ with $a < b$. Let $S = \{x \in A \colon x < b\}$. Note that S is nonempty and bounded above. Consider $\sup(S)$. Then $\sup(S) \in A$ or $\sup(S) \in B$. Try to get a contradiction in either case.

Proof. Suppose by way of contradiction that $\mathbb{R} = A \cup B$, where A and B are nonempty, disjoint open subset of $\mathbb{R}$. Without loss of generality, there exists $a \in A$ and $b \in B$ with $a < b$. Let $S = \{x \in A \colon x < b\}$. Note that S is nonempty and bounded above. Consider $\sup(S)$. Then $\sup(S) \in A$ or $\sup(S) \in B$. We will derive a contradiction in either case.

Case 1: $z = \sup(S) \in A$. Since A is open, there is some $\epsilon > 0$ such that $B_\epsilon(z) \subseteq A$. But this implies that there exists some $y \in S$ such that $y > z$, contradicting that z is an upper bound of S.

Case 2: $z = \sup(S) \in B$. Since B is open, there is some $\epsilon > 0$ such that $B_\epsilon(z) \subseteq B$. But then if $z - \epsilon < x \leq z$, then $x \in B$ but not in A (since A and B are disjoint).This contradicts z being the *least* upper bound of S. □

(5) Let n be a positive integer, and consider the function $f \colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^n$. You may assume that f is continuous. Use the Intermediate Value Theorem (along with some elementary reasoning) to show that $f([0, \infty)) = \{f(x) \colon x \in [0, \infty)\}$ ($\leftarrow$ is the definition of $f([0, \infty))$) $= [0, \infty)$. Conclude that for every real $r \geq 0$, there exists a real number $s \geq 0$ such that $s^n = r$.

Proof. Let $f \colon [0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^n$, where n is a positive integer. It is clear that $f(x) \geq 0$ for all $x \geq 0$. Moreover, if M is any non-negative real number, then $M < (M + 1)^n$ (the reason for the $M + 1$ is in case $0 < M < 1$; in this case, $M < M^n$ is false). It follows that, since $f([0, \infty))$ is an interval, we get $f([0, \infty)) = [0, \infty)$. So now let $r \in [0, \infty)$ be arbitrary. Then $r \in f([0, \infty))$, and hence $r = f(x)$ for some $x \in [0, \infty)$, that is, $r = x^n$. This completes the proof. □