

# Math 3410 Lecture 24

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Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = 1$  if  $x \geq 0$  and  $f(x) = -1$  if  $x < 0$ .

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Let  $f(x) = x^2$  for  $x \neq 0$  and let  $f(0) = 1$ . Then again, observe that you cannot draw the graph of  $f$  around 0 without lifting your pencil. Question: does  $\lim_{x \rightarrow 0} f(x)$  exist?

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# Continuity

We close this set of notes by proving that every continuous function on a closed bounded interval attains a global maximum. Similarly, every such function attains a global min, which I may leave to you to prove.

## Theorem

*Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f$  has a global maximum.*

## Proof.

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then as we have seen  $\{f(x): x \in [a, b]\}$  is bounded above. It is also clearly nonempty, and so the supremum of this set exists; call it  $M$ . We will prove that there is some  $d \in [a, b]$  such that  $f(d) = M$ . Suppose not. For every positive integer  $n$ , note that  $M - \frac{1}{n}$  is NOT an upper bound of  $\{f(x): x \in [a, b]\}$ . So we may choose some  $x_n \in [a, b]$  such that (\*)  $M - \frac{1}{n} < f(x_n) < M$ . As above, there is a monotonic subsequence  $(x_{n_k})$  which converges to some number  $x^* \in [a, b]$ . By continuity, we see that  $(f(x_{n_k})) \rightarrow f(x^*)$ . Note from above that we have  $M - \frac{1}{n_k} < f(x_{n_k}) < M$  for all  $k \in \mathbb{N}$ . □

# Continuity

Proof.

So now we have  $M - \frac{1}{n_k} < f(x_{n_k}) < M$  for all positive integers  $k$ .



# Continuity

Proof.

So now we have  $M - \frac{1}{n_k} < f(x_{n_k}) < M$  for all positive integers  $k$ . If we apply the Squeeze Theorem, we see that  $(f(x_{n_k})) \rightarrow M$ .

# Continuity

Proof.

So now we have  $M - \frac{1}{n_k} < f(x_{n_k}) < M$  for all positive integers  $k$ . If we apply the Squeeze Theorem, we see that  $(f(x_{n_k})) \rightarrow M$ . But recall above that  $(f(x_{n_k})) \rightarrow f(x^*)$ , and so we must have  $f(x^*) = M$ , as desired. □