Math 3410 Lecture 25

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So the elements of $B_\epsilon(r)$ are the real numbers which are a distance of less than ϵ from r. Again, remember (this is important!) that $|x - r| < \epsilon$ is equivalent to $-\epsilon < x - r < \epsilon$, that is, $r - \epsilon < x < r + \epsilon$.

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Let $S \subseteq \mathbb{R}$. Then we say that S is an **open set** provided that for every $s \in S$, there exists some $\epsilon > 0$ such that $B_\epsilon(s) \subseteq S$.

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Let $a \leq b$ be real, and assume by way of contradiction that $[a, b]$ is open.
Then there is some $\epsilon > 0$ such that $B_\epsilon(b) \subseteq [a, b]$, that is,
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The following are the real intervals:

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Let $S \subseteq \mathbb{R}$ then S is an interval if and only if for all $x, y, z \in \mathbb{R}$: if $x, z \in S$ and $x < y < z$, then $y \in S$.

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There are many cases to consider.

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Let $S \subseteq \mathbb{R}$, and suppose that S has the property that (*) for all $x, y, z \in \mathbb{R}$: if $x, z \in S$ and $x < y < z$, then $y \in S$. We will show that S is an interval. There are many cases to consider.

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Let f,I, and O be as stated. Now let $c \in f^{-1}(O)$. Then $f(c) \in O$, which is open. Thus there exists some $\epsilon > 0$ such that $B_\epsilon(f(c)) \subseteq O$.

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Lemma

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Let I be an interval. Then I cannot be expressed as the union of two disjoint, nonempty open subset of $\mathbb{R}$.

Proof.

Exercise. $\square$

IVT

Theorem (Intermediate Value Theorem)

Suppose that I is an interval and $f : I \to \mathbb{R}$ is a continuous function.

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