Math 3110 Assignment 5 Solutions

(1)[10 pts] Let a be a positive integer. What is $\gcd(a, 2a)$? Prove your assertion.

Proof. Let a be a positive integer. Then $\gcd(a, 2a) = a$. To see this, note that since $a \cdot 1 = a$ and $a \cdot 2 = 2a$, $a|a$ and $a|2a$. Now supose that $x \geq a$ is an integer such that $x|a$ and $x|2a$. Then since a is positive, so is x, since $x \geq a$. So now x and a are positive. Since $x|a$, $x \leq a$, and this shows that (since from above, $x \geq a$) $x = a$, proving that a is the gcd. You can also use the Char. of the GCD Thm. here. ☐

(2)[10 pts] Let a be a positive integer. What is $\operatorname{lcm}(a, 2a)$? Prove your assertion. Remember, this means showing that your conjectured least common multiple is positive, is a multiple of both a and $2a$, and is the smallest positive such integer.

Proof. Let a be a positive integer. Then $\operatorname{lcm}(a, 2a) = 2a$. To see this, note that $a \cdot 2 = 2a$ and $2a \cdot 1 = 2a$. Thus $2a$ is a common multiple of a and $2a$. Since a is positive, so is $2a$. It remains to show that $2a$ is the least common positive multiple of a and $2a$. Thus, let x be an arbitrary positive, common multiple of a and $2a$. Since $2a|x$ and both are positive, $2a \leq x$, showing that $2a$ is the least positive common multiple of a and $2a$. ☐

(3)[5 pts] (this is will be worth VERY few points) What is, for a positive integer a, $\gcd(a, 2a) \cdot \operatorname{lcm}(a, 2a)$? You do NOT need to justify your answer.

Solution. $\gcd(a, 2a) \cdot \operatorname{lcm}(a, 2a) = 2a^2$. ☐

(4)[15 pts] Let a and b be positive integers, and let c be a positive integer such that $c|a$ and $c|b$. Prove that $\text{lcm}(\frac{a}{c}, \frac{b}{c}) = \frac{\text{lcm}(a,b)}{c}$. Hint: let $m = \text{lcm}(\frac{a}{c}, \frac{b}{c})$. You need to prove that $mc = \text{lcm}(a, b)$. Can you show that (a) $mc > 0$, (b) $a|mc$, $b|mc$, and (c) for every integer $x > 0$, if $a|x$ and $b|x$, then $mc \leq x$? Try to break the proof into these pieces, work out on scratch paper, then construct the proof.

Proof. Let a and b be positive integers, and let c be a positive integer such that $c|a$ and $c|b$. We will prove that $\text{lcm}(\frac{a}{c}, \frac{b}{c}) = \frac{\text{lcm}(a,b)}{c}$. Toward this end, let $m = \text{lcm}(\frac{a}{c}, \frac{b}{c})$. So it now suffice to prove that $m = \frac{\text{lcm}(a,b)}{c}$, that is, $mc = \text{lcm}(a, b)$. So we have several things to prove.

(i) $mc > 0$: this is immediate, since by definition of 'lcm', $m > 0$ and we are given that $c > 0$. Thus $mc > 0$.

(ii) $a|mc$ and $b|mc$: to see this, recall above that $m = \text{lcm}(\frac{a}{c}, \frac{b}{c})$. Thus $\frac{a}{c}|m$ and $\frac{b}{c}|m$. This means that there are integers x and y such that $\frac{a}{c}x = m$ and $\frac{b}{c}y = m$. Multiplying through by c in both equations gives $ax = mc$ and $by = mc$. Thus $a|mc$ and $b|mc$.

(iii) Let z be a POSITIVE common multiple of both a and b. We must show that $mc \leq z$. Since $a|z$ and $b|z$, we see that there are integers u and v such that $au = bv = z$. Now, since $c|a$ and $a|z$, $c|z$ (divisibility theorem); thus $\frac{z}{c}$ is a POSITIVE INTEGER. From $au = z$, we get $\frac{a}{c}u = \frac{z}{c}$. Similarly, from $bv = z$, we get $\frac{b}{c}v = \frac{z}{c}$. Thus $\frac{z}{c}$ is a common positive multiple of $\frac{a}{c}$ and $\frac{b}{c}$. Since m is the LEAST common multiple of $\frac{a}{c}$ and $\frac{b}{c}$, we must have $m \leq \frac{z}{c}$. Multiplying by c, we get $mc \leq z$, as desired. □