Math 3110 Assignment 7 Solutions

(1)[5 pts] Prove by cases that for every real number r, we have $r^2 = |r|^2$.

Proof. Let r be an arbitrary real number. Then either $r \geq 0$ or $r < 0$, giving us cases.

Case 1: $r \geq 0$. Then $|r| = r$, and so $r^2 = |r|^2$ in this case.

Case 2: $r < 0$. Then $|r| = -r$, and so $r^2 = (-r)^2 = |r|^2$. □

(2)[5 pts] Remember that an integer m is a **perfect square** provided $m = x^2$ for some *integer* x (note that I didn't say x is positive). Use (1) to prove that if $m > 1$ is a perfect square, then $m = n^2$ for some integer n such that $n > 1$.

Proof. Let $m > 1$ be an integer, and assume that m is a perfect square. Then $m = x^2 = |x|^2$ for some integer x (the final equation follows from problem (1)). Now, $|x| \geq 0$ is an integer, so we must simply rule out $|x| = 0$ and $|x| = 1$. Notice that if $|x| = 0$ or $|x| = 1$, then $m = |x|^2 = 0$ or $m = |x|^2 = 1$, contradicting that $m > 1$. □

(3)Let $m > 1$ be an integer, and let (via the FTA) $m = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of m in canonical form, where $p_1 < p_2 < \cdots < p_k$ are primes and each a_i is a positive integer.

(a)[5 pts] If each a_i is even, prove that m is a perfect square. This is essentially just exponent rules/basic algebra, and the proof is not long.

Proof. Let $m = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, with the p_i and a_j as stated above. Assume that each a_i is even. Then for $1 \leq i \leq k$, we have $a_i = 2b_i$ for some positive integer b_i (can you see why b_i is positive?). Hence $m = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k} = p_1^{2b_1} \cdot p_2^{2b_2} \cdots p_k^{2b_k} = (p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k})^2$, showing that m is a perfect square. □

(b)[10 pts] Prove that if m is a perfect square, then each a_i is even. Hint: by (2), $m = n^2$ for some integer $n > 1$. So n has a prime factorization in canonical form. Use this along with the *uniqueness* part of the FTA to deduce that each a_i is even.

Proof. With m, p_i, and a_j as stated above, assume that m is a perfect square. By (2), $m = n^2$ for some integer $n > 1$. By the FTA, we see that $n = q_1^{b_1} \cdots q_s^{b_s}$, where this product is in canonical form. Squaring, we get $p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k} = m = n^2 = q_1^{2b_1} \cdots q_s^{2b_s}$. By the uniqueness component of the FTA, we see that $k = s$ and for $1 \leq i \leq k$, we have $a_i = 2b_i$, showing that each a_i is even. □

(4a)[6 pts] Recall that a real number r is **rational** provided $r = \frac{a}{b}$ for some integers a and b such that $b \neq 0$. Prove that if $r > 0$ is a rational number, then there are integers a and b with $a, b > 1$ such that $r = \frac{a}{b}$.

Proof. Let $r > 0$ be a rational number. Then $r = \frac{c}{d}$ for some integers c and d for which $d \neq 0$. Note that also $c \neq 0$, lest $r = 0$. Moreover, we cannot have exactly one of c, d positive and the other negative, lest $r < 0$. So we know that c and d are either both positive or both negative. This leads us to cases.

Case 1: c and d are both positive. Then $r = \frac{c}{d} = \frac{2c}{2d}$, and both $2c, 2d > 1$.

Case 2: c and d are both negative. Then $-c$ and $-d$ are both positive. Moreover, $r = \frac{c}{d} = \frac{-c}{-d}$. Now apply Case 1. □

(4b)[5 pts - completion only] Use the previous results of this hw to prove that if $m > 1$ is an integer such that $\sqrt{m}$ is rational, then m is a perfect square.

Proof. Let $m > 1$ be an integer such that $\sqrt{m}$ is rational. We will prove that m is a perfect square. Then from our work above, we have $\sqrt{m} = \frac{a}{b}$ for some integers a and b which are both greater than 1. Squaring and clearing fractions, we get $mb^2 = a^2$. Now, we have $m = p_1^{r_1} \cdots p_k^{r_k}$, $b^2 = q_1^{2s_1} \cdots q_l^{2s_l}$, and $a^2 = v_1^{2e_1} \cdots v_j^{2e_j}$, where are products are products of primes in canonical form. Since $mb^2 = a^2$, we get $p_1^{r_1} \cdots p_k^{r_k} q_1^{2s_1} \cdots q_l^{2s_l} = v_1^{2e_1} \cdots v_j^{2e_j}$. Now, the right-hand side of the equation above is in canonical form. Upon putting the left-hand side into canonical form, if any r_i is odd, then p_i will occur to an odd power on the left-hand side of the equation, but to an even power on the right (by the FTA), a contradiction. Thus each r_i is even, and so from problem 3a), m is a perfect square. □