FROM ANCIENT EGYPTIAN FRACTIONS TO MODERN ALGEBRA

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Abstract. An *Egyptian fraction* is a finite sum of distinct rational numbers of the form $\frac{1}{m}$, where m is a nonzero integer. It is well-known that every rational number can be expressed as an Egyptian fraction. The purpose of this note is to explore natural analogs of this concept for commutative integral domains.

1. Introduction

1.1. **History.** Suppose that 8 friends are hungry and decide to buy 5 pizzas to be split equally among themselves. How might one divide them up? There are 8 people total, so one way to do this is to cut each pizza into 8 equal slices, and then give each person 5 slices. But is there a simpler way to accomplish this? Yes, there is. Notice that $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$. Thus one can, equivalently, give each person $\frac{1}{2}$ of a pizza and $\frac{1}{8}$ of a pizza. Note that the latter solution enables one to make fewer cuts.[1]

Problems such as the one presented above date back to antiquity. Indeed, the ancient Egyptians did not express rational numbers the way we do today; they had no notation for $\frac{5}{8}$. They only had notation for so-called *unit fractions*, that is, rational numbers with numerator 1. As the first paragraph demonstrates, there are contexts in which expressing a rational number as a sum of units fractions enables one to find more elegant solutions to certain problems.

Now, it is clear that every rational number may be expressed as a sum of unit fractions; for example, $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$. But also, observe that $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$. The convention adopted by the Egyptians (circa 3000 B.C.) was to always represent a rational number as the sum of *distinct* unit fractions; such a representation is called an *Egyptian fraction*. This raises the following natural question: can every rational number be expressed as an Egyptian fraction? It turns out that the answer is yes. This was shown by Fibonacci in his 1202 book *Liber Abaci* (though it may have been proven well before this); see [5] for an English translation of his work. For elementary applications and some different proofs of this result, we refer the reader to [11], [13], and [14].

1.2. **Motivation.** Egyptian fractions have proven to be more important to pure mathematics than one may guess from our pizza-cutting example above. For instance, the question of whether any positive real number is the groupoid cardinality of some groupoid reduces to the question of whether any positive rational number has an Egyptian fraction decomposition ([1]). Moreover, several questions and conjectures concerning Egyptian fractions have been solved in modern times (see [4] and [8]), and yet others persist. Possibly the best-known open problem in the area is the so-called Erdós-Straus conjecture, which asserts that for every integer $n \geq 2$, there exist distinct positive

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[1]This example is due to Ron Knott.

integers x, y, and z for which $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. For further background on this and other problems in number theory, we refer the reader to [7].

The purpose of this paper is to translate the notion of "Egyptian fraction" to the more general setting of abstract algebra, specifically, to integral domains. We define an integral domain D to be an *Egyptian domain* provided that every nonzero $d \in D$, we have $d = \frac{1}{d_1} + \cdots + \frac{1}{d_k}$ for some nonzero, distinct $d_1, \ldots, d_k \in D$, where each $\frac{1}{d_i}$ is a member of the quotient field of D (we will develop this definition in the next section). Our motivation is to introduce this definition, prove some fundamental results, and entice others to continue our investigations.

2. Main Results

2.1. **Egyptian fractions in the rational numbers.** Our first task is to establish the well-known result that every rational number can be represented by an Egyptian fraction, that is, as a sum of unit fractions with distinct denominators. Our proof is based on a greedy algorithm used by Fibonacci.

Theorem 1 (Fibonacci)**.** *Every rational number can be represented by an Egyptian fraction. Moreover, for every positive rational number r and positive integer n, one can represent r as an Egyptian fraction all of whose denominators are greater than n.*

Proof. Note first that 0 can be so represented as follows: $0 = \frac{1}{1} + \frac{1}{-1}$. If one desires denominators with distinct magnitudes, observe that $0 = \frac{1}{2} - \frac{1}{3} - \frac{1}{6}$.

Next, if we can show that every positive rational number has an Egyptian representation, then by taking negatives, we immediately get that every negative rational number has such a representation as well. So it remains to show that every positive rational number has an Egyptian representation (as we will see, we may choose the representation to consist of positive rational numbers).

We first establish that

(2.1) every positive rational number less than one has an Egyptian fraction representation.

Indeed, consider a rational number $r := \frac{p}{q}$ with $0 < p < q$. If $p = 1$, then r is a unit fraction and we are done. So assume that $p > 1$. Next, choose the least positive integer n such that

$$\frac{1}{n} < \frac{p}{q}, \tag{2.2}$$

and observe that $n > 1$. By minimality of n, we deduce that $\frac{1}{n-1} \geq \frac{p}{q}$. If $\frac{1}{n-1} = \frac{p}{q}$, then we are done, and so we may assume that

$$\frac{1}{n-1} > \frac{p}{q}. \tag{2.3}$$

Now, from (2.2), we see that

$$\frac{p}{q} - \frac{1}{n} = \frac{np-q}{qn} > 0, \tag{2.4}$$

from which it follows that $np-q > 0$. On the other hand, (2.3) yields that $q > p(n-1)$, from which it follows that $np-q < p$. So we have shown that $0 < np-q < p$. If $np-q = 1$, then from (2.4), we have $\frac{p}{q} = \frac{1}{n} + \frac{1}{qn}$. Recall that $0 < p < q$, and so $q > 1$. We deduce that $n < qn$; this expresses $\frac{p}{q}$ as an Egyptian fraction. So suppose that $1 < np-q < p$. Then we have $0 < np-q < p < q < qn$ (recall that $n > 1$). We now repeat the process above for the fraction $\frac{np-q}{qn}$, which has numerator strictly less than p. Continuing this process recursively, the process must terminate in a unit fraction. One shows easily by induction that whenever the process terminates, this algorithm represents $\frac{p}{q}$ as a sum of *distinct* (positive) unit fractions.

We now complete the proof. Consider a positive rational number r and let n be a positive integer such that $\frac{1}{n} < r$. Next, choose the largest positive integer k such that

$$\frac{1}{n+1} + \cdots + \frac{1}{n+k} \leq r. \tag{2.5}$$

If equality holds, then we have a representation of r as an Egyptian fraction. So let us assume that $\frac{1}{n+1} + \cdots + \frac{1}{n+k} < r$. Invoking the maximality of k, there is a rational number s, $0 < s < 1$, for which $\frac{1}{n+1} + \cdots + \frac{1}{n+k} + s = r$. From what we proved above, s has an Egyptian fraction decomposition as a sum of distinct positive unit fractions. It is clear by maximality of k that each denominator in the Egyptian representation of s has denominator larger than $n+j$ for $1 \leq j \leq k$, and the proof is complete. □

2.2. **Generalizing to rings: definitions and examples.** Now that we have established that every rational number has an Egyptian fraction decomposition, we work toward generalizing. Toward this end, we make the following elementary observation: suppose we have shown that every integer has a representation as an Egyptian fraction. Then every rational number can also be so represented: consider a rational number of the form $\frac{p}{q}$. Begin by expressing p as an Egyptian fraction, and then simply multiply the representation by $\frac{1}{q}$. Upon distributing, we get an Egyptian representation for $\frac{p}{q}$. This motivates us to make the following definition.

Definition 1. Let D be an (unital) integral domain. Call D an ***Egyptian domain*** provided that for every nonzero[2] $d \in D$, there exist distinct nonzero $d_1, \ldots d_n \in D$ such that $d = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$, where the unit fractions belong to the quotient field of D.

It follows from our work above that the ring $\mathbb{Z}$ of integers is an Egyptian domain. We now present several additional examples.

Example 1. *Every field is an Egyptian domain.*

[2]We will explain shortly why we exclude zero.

Proof. Let F be a field, and let $\alpha \in F\backslash\{0\}$. Then simply note that $\alpha = \frac{1}{\alpha^{-1}}$, from which it follows that F is Egyptian. $\square$

Remark 1. In light of the above example, we pause to explain why we require only the *nonzero* elements of a domain to have an Egyptian representation. If we require 0 to be so represented, then observe that $0 \in \mathbb{F}_2$ (the field with two elements) has no Egyptian representation.

Example 2. *The ring* $\mathbb{Z}[\sqrt{2}]$ *is Egyptian.*

Proof. Consider an arbitrary $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ with a and b integers which are not both zero. Now rationalize the numerator to obtain $a + b\sqrt{2} = \frac{a^2-2b^2}{a-b\sqrt{2}}$. We may express the numerator as an Egyptian fraction by Theorem 1. Now simply multiply the expression by $\frac{1}{a-b\sqrt{2}}$ to obtain an Egyptian form for $a + b\sqrt{2}$. $\square$

Observe that the examples above are all Jacobson semisimple (that is, the Jacobson radical is trivial). It turns out that every domain with nonzero Jacobson radical is Egyptian, as we now prove.

Example 3. *Let D be a domain. If the Jacobson radical of D is nonzero, then D is Egyptian. In this case, every nonzero $d \in D$ may be expressed as a sum of no more than three distinct unit fractions.*

Proof. Let D be a domain with nonzero Jacobson radical $J(D)$, and let $d \in D\backslash\{0\}$ be arbitrary. Now choose a nonzero $j \in J(D)$. Observe that $dj - 1 := u$ is a unit of D. Solving for d, we get $d = \frac{u}{j} + \frac{1}{j} = \frac{1}{u^{-1}j} + \frac{1}{j}$. If $u^{-1}j$ and j are distinct, then we are done. So suppose $u^{-1}j = j$. Then $u = 1$. But now $d = \frac{1}{j} + \frac{1}{j} = \frac{1}{j} + \frac{1}{j+1} + \frac{1}{j(j+1)}$. Recall that $j \in J(D)$ was chosen to be nonzero. Moreover, $j + 1 \neq 0$, lest $-1 \in J(D)$, which is absurd since $J(D)$ is a proper ideal of D. It follows that the denominators above are nonzero, and it is easy to check that they are distinct, concluding the proof. $\square$

Corollary 1. *Every semilocal domain is Egyptian.*

Proof. Let D be a semilocal domain. If D is a field, we invoke Example 1. Now assume that D is not a field, and let $M_1, \ldots, M_n$ be the maximal ideals of D. Because D is not a field, no M_i is trivial. Choose a nonzero $m_i \in M_i$ for each i. Then $m_1 \cdots m_n \in J(D)\backslash\{0\}$. $\square$

We conclude this subsection with the following example.

Example 4. *Let D be a domain, and let S be the set of all nonzero elements of D which cannot be expressed as an Egyptian fraction. For each $s \in S$, introduce an indeterminate X_s. Now let $D^* := D[X_s, \frac{1}{X_s}, \frac{1}{X_s - s} : s \in S]$. For any $s \in S$, observe that $\frac{1}{\frac{1}{X_s}} - \frac{1}{\frac{1}{X_s - s}} = s$. Thus every member of D has an Egyptian representation in D^*, though D^* may have elements with no Egyptian representation. So continue recursively, that is, set $D_0 := D$ and for every non-negative integer n, let $D_{n+1} := (D_n)^*$. Now let $D_\infty := \bigcup_{n\in\mathbb{N}} D_n$. Then D_∞ is an Egyptian domain containing D. Moreover, $D_\infty \cap K = D$, where K is the quotient field of D.*

2.3. **Behavior relative to ring constructions.** The purpose of this subsection is to study ring constructions and determine which preserve the Egyptian property. We begin with a negative result.

Proposition 1. *Let D be a domain. The polynomial ring $D[X]$ is not Egyptian.*

Proof. Suppose by way of contradiction that there is a domain D such that $D[X]$ is Egyptian. It follows that $X = \frac{1}{f_1(X)} + \cdots + \frac{1}{f_n(X)}$ for some nonzero $f_1(X), \ldots, f_n(X) \in D[X]$. Now multiply through by $f_1(X) \cdots f_n(X)$ to obtain $Xf_1(X) \cdots f_n(X)$ as the sum of monomials, each of degree at most $\deg(f_1(X) \cdots f_n(X))$. But then the degree of $Xf_1(X) \cdots f_n(X)$ is at most $\deg(f_1(X) \cdots f_n(X))$, which is absurd. □

Taking D to be a field in the previous proposition, we see that there exist examples of Noetherian non-Egyptian domains. Indeed, there exist Euclidean domains which are not Egyptian. We can now show that the Egyptian property is not inherited by subrings: take any domain D and consider the field $D(X)$ of rational functions in X with coefficients in D. Then $D(X)$ is Egyptian by Example 1, but the subring $D[X]$ is not.

We now show that power series rings are much better behaved.

Proposition 2. *If D is a domain, then the power series ring $D[[X]]$ is Egyptian.*

Proof. Let D be a domain. It is well-known that $f(X) \in D[[X]]$ is a unit if and only if the constant term of $f(X)$ is a unit in D.[3] This implies that for any $f(X) \in D[[X]]$, $Xf(X) - 1$ is a unit of $D[[X]]$. Thus $X \in J(D[[X]])$, showing that $D[[X]]$ is not Jacobson semisimple. Thus by Example 3, $D[[X]]$ is Egyptian. □

The previous propositions enable us to show that the Egyptain property is not preserved by homomorphic images.

Corollary 2. *Let D be a non-Egyptian domain. Then $D[[X]]$ is Egptian, but $D[[X]]/\langle X\rangle \cong D$ is not.*

Next, we recalll that if R is a ring and G is a group, then the ***group ring*** $R[G]$ consists of all finitely nonzero formal sums Σrg for $r \in R, g \in G$ with addition and multiplication defined analogously to polynomial rings (monomial multiplication is defined by $(r_1g_1) \cdot (r_2g_2) := (r_1r_2)(g_1g_2)$). It is known that if R is a unital (possibly noncommutative) ring and G is a group, then the group ring $R[G]$ is prime if and only if R is a prime ring and G has no finite, nontrivial normal subgroups (see [3]). In case R and G are commutative, this translates to $R[G]$ being an integral domain if and only if R is a domain and G is torsion-free. In this setting as well, the Egyptian property easily transfers.

Proposition 3. *Let D be an Egyptian domain and G be a torsion-free abelian group. Then $D[G]$ is Egyptian.*

[3]For this result and for further reading on power series rings, see [7].

Proof. Suppose D is an Egyptian domain and G is a torsion-free abelian group. Consider a nonzero element of $D[G]$, say $d_1g_1+\cdots d_ng_n$, where the d_i are nonzero and the g_i are distinct. For $1 \leq i \leq n$, express d_i as an Egyptian fraction in D and then multiply by $\frac{1}{g_i^{-1}}$. The resulting sum yields an Egyptian representation of $d_1g_1+\cdots+d_ng_n$. □

The Egyptian condition also passes to overrings, which is similarly easy to prove.

Proposition 4. *Suppose that D is an Egyptian domain with quotient field K. If D' is a domain such that $D \subseteq D' \subseteq K$, then D' is also Egyptian.*

Proof. Let D, D', and K be as stated. Now let $\frac{a}{b} \in D'$, where a and b are nonzero elements of D. Since D is Egyptian, we may express a as the sum of distinct unit fractions with denominators in D. Now simply multiply both sides of this equation by $\frac{1}{b}$ and distribute. □

Remark 2. If D is a domain, recall that the ring of ***integer-valued polynomials***, $\mathrm{Int}(D)$, is defined by $\mathrm{Int}(D) := \{f(X) \in K[X] \colon f(D) \subseteq D\}$, where K is the quotient field of D. If $\mathrm{Int}(D)$ were Egyptian, then by Proposition 4, every overring of $\mathrm{Int}(D)$ would be Egyptian. It is immediate that $D[X] \subseteq \mathrm{Int}(D)$, and thus $K[X]$ is an overring of $\mathrm{Int}(D)$. But by Proposition 1, $K[X]$ is not Egyptian, and this contradicts Proposition 4. Thus $\mathrm{Int}(D)$ is never Egyptian.

We will shortly present a partially negative result, but we first show that the Egyptian property is preserved under algebraic extensions.

Proposition 5. *Let $D \subseteq T$ be domains, with T is algebraic over D. If D is Egyptian, so is T. The converse fails.*

Proof. Suppose that $D \subseteq T$ is an algebraic extension of domains with D Egyptian, and suppose $x \in T\backslash\{0\}$. We will prove that x has an Egyptian representation. Since x is algebraic over D, there is a nonzero $d \in D$ for which $t := dx$ is integral over D. Thus

$$t^n + d_{n-1}t^{n-1} + \cdots + d_1t + d_0 = 0 \tag{2.6}$$

for some $d_0, \ldots, d_{n-1} \in D$ and with n minimal. Now solve (2.6) for d_0 to obtain

$$d_0 = -d_1t - \cdots - t^n. \tag{2.7}$$

Note that by minimality of n, we have

$$d_0 \neq 0. \tag{2.8}$$

Now divide by sides by t (recall above that $t \neq 0$) to get

$$\frac{d_0}{t} = -d_1 - d_2t - \cdots - t^{n-1} := t_0 \in T. \tag{2.9}$$

Observe that $t_0 \neq 0$, lest $d_0 = 0$, contradicting (2.8) above. So now we have $\frac{d_0}{t} = t_0$. Solving for t gives $t = \frac{d_0}{t_0} = \frac{1}{t_0}d_0$. Express d_0 as a sum of distinct unit fractions, and then multiply both sides by $\frac{1}{t_0}$ to obtain an Egyptian expression for t. Recall above that $t = dx$. So simply divide through by d to obtain an Egyptian expression for x.

To see that the converse fails, let D be any non-Egyptian domain with quotient field K. By Example 1, K is Egyptian. Moreover, K is clearly algebraic over D. But D is not Egyptian. $\square$

Remark 3. It follows from Theorem 1 and Proposition 5 that every ring of algebraic integers is Egyptian.

Our next goal is to study the Egyptian property relative to ultraproducts. The reader not familiar with this construction is encouraged to consult [2] and [10]. However, we shall make what follows mostly self-contained (though somewhat terse).

Definition 2. Let S be a nonempty set. An ***ultrafilter*** on S is a collection $\mathcal{U}$ of subsets of S which satisfies the following conditions:

(A1) $\varnothing \notin \mathcal{U}$,
(A2) $\mathcal{U}$ is closed under supersets,
(A3) $\mathcal{U}$ is closed under finite intersections, and
(A4) For any subset X of S, either $X \in \mathcal{U}$ or $S \backslash X \in \mathcal{U}$.

We encourage the reader to verify that the following is an example of an ultrafilter.

Example 5. *Let S be a nonempty set and let $s \in S$. Now let $\mathcal{U} := \{X \subseteq S : s \in X\}$. Then $\mathcal{U}$ is an ultrafilter on S, called a* ***principal ultrafilter****.*

Using the axiom of choice, one can show that for every infinite set S, there exists a non-principal ultrafilter on S. We will shortly take advantage of this fact. First, we give the definition of an ultraproduct of a collection of commutative rings.

Definition 3. Let $\{R_i : i \in I\}$ be a nonempty collection of rings enumerated by the index set I. Further, suppose that $\mathcal{U}$ is an ultrafilter on I. We now define the following relation on the direct product $\Pi_{i\in I} R_i$ as follows: $(r_i) \sim (s_i)$ if and only if $\{i \in I : r_i = s_i\} \in \mathcal{U}$. It is not hard to verify that $\sim$ is an equivalence relation compatible with the ring operations. The ring $\prod_{i\in I} R_i/\sim$ is called the ***ultraproduct*** of the rings R_i. If $R_i = R$ for all $i \in I$, then the ultraproduct is often called an ***ultrapower*** of R. Let us denote the ultraproduct of the R_i (with respect to $\mathcal{U}$) by $(\Pi_{i\in I} R_i)_{\mathcal{U}}$ and an element of the ultraproduct by $(r_i)_{\mathcal{U}}$ (to distinguish between the ultraproduct and the direct product).

We will shortly prove that, unlike the direct product, an ultraproduct of integral domains remains an integral domain. To do this, we shall make use of the following (well-known) lemma.

Lemma 1. *Let S be a nonempty set, and suppose that $\mathcal{U}$ is an ultrafilter on S. Then the following hold.*

(1) *For all $n \in \mathbb{Z}^+$: if $A_1, \ldots, A_n$ are subsets of S such that $A_1 \cup \cdots \cup A_n \in \mathcal{U}$, then $A_i \in \mathcal{U}$ for some i.*
(2) *$\mathcal{U}$ is principal if and only if $\mathcal{U}$ contains some finite set.*

Proof. Suppose that S and $\mathcal{U}$ are as stated.

(1) By induction, it clearly suffice to prove that if A and B are subsets of S such that $A \cup B \in \mathcal{U}$, then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$. Suppose by way of contradiction that there are subsets A and B of S such that $A \cup B \in \mathcal{U}$, yet $A \notin \mathcal{U}$ and $B \notin \mathcal{U}$. By (A4), A^c and B^c are members of $\mathcal{U}$, and so by (A3), $A^c \cap B^c = (A \cup B)^c \in \mathcal{U}$. Invoking (A3) again, $(A \cup B) \cap (A \cup B)^c = \varnothing \in \mathcal{U}$, contradicting (A1).

(2) Thie forward implication is clear. As for the reverse implication, suppose that $\mathcal{U}$ contains a finite set. By (A1), this finite set is nonempty. Applying (1), $\{s\} \in \mathcal{U}$ for some $s \in S$. By (A2), $\mathcal{U}$ contains all subsets of S which contain s. Suppose by way of contradiction that there exists $A \in \mathcal{U}$ which does not contain s. Applying (A3), $A \cap \{s\} = \varnothing \in \mathcal{U}$, contradicting (A1). $\square$

We can now show that an ultraproduct of integral domains remains a domain (this is well-known, of course).

Lemma 2. *An ultraproduct of a collection of integral domains is an integral domain.*

Proof. Let $\{D_i : i \in I\}$ be a nonempty collection of integral domains, and let $\mathcal{U}$ be an ultrafilter on I. Let $(a_i)_{\mathcal{U}}, (b_i)_{\mathcal{U}} \in (\prod_{i \in I} D_i)_{\mathcal{U}}$, and suppose that $(a_i)_{\mathcal{U}}(b_i)_{\mathcal{U}} = (0)_{\mathcal{U}}$. Then (by definition) $(a_i b_i)_{\mathcal{U}} = (0)_{\mathcal{U}}$, and it follows that $C := \{i \in I : a_i b_i = 0\} \in \mathcal{U}$. Let $A := \{i \in I : a_i = 0\}$ and $B := \{i \in I : b_i = 0\}$, and note that $A \cup B = C$. By Lemma 1(1), either $A \in \mathcal{U}$ or $B \in \mathcal{U}$, implying that either $(a_i)_{\mathcal{U}} = (0)_{\mathcal{U}}$ or $(b_i)_{\mathcal{U}} = (0)_{\mathcal{U}}$. This completes the argument. $\square$

Next, we prove that the Egyptian property does not pass to ultraproducts in general.

Proposition 6. *Consider the direct product $\Pi_{n \in \mathbb{Z}^+} \mathbb{Z}$ of ω copies of the ring $\mathbb{Z}$ of integers, and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{Z}^+$. Then the ultrapower $(\Pi_{n \in \mathbb{Z}^+} \mathbb{Z})_{\mathcal{U}}$ is not Egyptian.*

Proof. Let $D := (\Pi_{n \in \mathbb{Z}^+} \mathbb{Z})_{\mathcal{U}}$, where $\mathcal{U}$ is a non-principal ultrafilter on the set of positive integers. Now consider the element $(1, 2, 3, \ldots)_{\mathcal{U}} := (n)_{\mathcal{U}}$ of D. Since $\varnothing \notin \mathcal{U}$, it follows that $(n)_{\mathcal{U}}$ is a nonzero element of D. We claim that $(n)_{\mathcal{U}}$ is not a finite sum of unit fractions (distinct or not). For suppose by way of contradiction that

$$(n)_{\mathcal{U}} = \frac{1}{(x_{1,n})_{\mathcal{U}}} + \cdots + \frac{1}{(x_{k,n})_{\mathcal{U}}} \tag{2.10}$$

for some nonzero $(x_{i.n})_{\mathcal{U}} \in D$, $1 \leq i \leq k$. Clearing the fractions, we obtain

$$(n x_{1,n} x_{2,n} \cdots x_{k,n})_{\mathcal{U}} = \left(\sum_{1 \leq i \leq k} (x_{1,n} x_{2,n} \cdots \cancel{x_{i,n}} \cdots x_{k,n}) \right)_{\mathcal{U}}. \tag{2.11}$$

Our next claim is that

$$\text{for all } n > k, \; x_{i,n} = 0 \text{ for some } i, \; 1 \leq i \leq k. \tag{2.12}$$

Indeed, suppose that $n > k$ and that $x_{i,n} \neq 0$ for all i, $1 \leq i \leq k$. Now observe that we have

$$
\begin{aligned}
\left|\sum_{1\le i\le k} x_{1,n}x_{2,n}\cdots \cancel{x_{i,n}}\cdots x_{k,n}\right| &\le \sum_{1\le i\le k} |x_{1,n}x_{2,n}\cdots \cancel{x_{i,n}}\cdots x_{k,n}| \\
&\le \sum_{1\le i\le k} |x_{1,n}x_{2,n}\cdots x_{k,n}| \\
&= |kx_{1,n}x_{2,n}\cdots x_{k,n}| \\
&< |nx_{1,n}x_{2,n}\ldots x_{k.n}|,
\end{aligned} \tag{2.13}
$$

from which it follows that $nx_{1,n}x_{2,n}\ldots x_{k.n} \neq \sum_{1\le i\le k} x_{1,n}x_{2,n}\cdots \cancel{x_{i,n}}\cdots x_{k,n}$. Said another way, we see that

$$
\text{if } nx_{1,n}x_{2,n}\ldots x_{k.n} = \sum_{1\le i\le k} x_{1,n}x_{2,n}\cdots \cancel{x_{i,n}}\cdots x_{k,n}, \text{ then } n \le k. \tag{2.14}
$$

But then $A := \{n \in \mathbb{Z}^+ \colon nx_{1,n}x_{2,n}\ldots x_{k.n} = \sum_{1\le i\le k} x_{1,n}x_{2,n}\cdots \cancel{x_{i,n}}\cdots x_{k,n}\}$ is a finite set which, by (2.11), is a member of $\mathcal{U}$. By Lemma 1(2), $\mathcal{U}$ is principal, a contradiction; this establishes (2.12). Now let $A := \mathbb{N}\backslash\{1,\ldots,k\}$. Invoking Lemma 1(2), (A4), and the fact that $\mathcal{U}$ is nonprincipal, it follows that $A \in \mathcal{U}$. Set $B := \{n \in \mathbb{Z}^+ \colon x_{i,n} = 0 \text{ for some } i, 1 \le i \le k\}$. It is clear from (2.12) that $A \subseteq B$, and so by (A2), $B \in \mathcal{U}$. For $1 \le i \le k$, set $B_i := \{n \in \mathbb{Z}^+ \colon x_{i,n} = 0\}$. Then note that $B = B_1 \cup \cdots \cup B_k$. Lemma 1(1) shows that $B_i \in \mathcal{U}$ for some i. But then $(x_{i.n})_{\mathcal{U}} = (0)_{\mathcal{U}}$, and this is a contradiction. $\square$

Some comments are now in order. A famous theorem of Łos in model theory asserts that in the language of commutative rings with identity, a (first-order) sentence φ is true in an ultraproduct $(\Pi_{i\in I}R_i)_{\mathcal{U}}$ if and only if the set $\{i \in I \colon \varphi \text{ is true in } R_i\}$ is a member of $\mathcal{U}$ (this holds more generally for models in an arbitrary first-order language, but we won't need this more general fact; for details, see [2]). Combined with Proposition 6 above, this shows that we cannot express the Egyptian property in the language of unital rings. More precisely, we have

Corollary 3. *Let* $\mathbf{L}$ *be the language of unital rings. There is no set* Σ *of* $\mathbf{L}$*-sentences such that for all* $\mathbf{L}$*-structures* $\mathfrak{M}$*,* $\mathfrak{M}$ *is an (unital, commutative) Egyptian domain if and only* $\mathfrak{M}$ *is a model of all sentences in* Σ*.*

Proof. Suppose such a collection Σ of sentences exists in the language of unital rings. Then the ring $\mathbb{Z}$ of integers is a model of Σ. By Łos' theorem, so is $(\prod_{n\in\mathbb{Z}^+}\mathbb{Z})_{\mathcal{U}}$, where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{Z}^+$. But then $(\prod_{n\in\mathbb{Z}^+}\mathbb{Z})_{\mathcal{U}}$ is also Egyptian, which contradicts Proposition 6. $\square$

Despite the negative results above, we can salvage a result in the positive direction. Indeed, consider an ultraproduct $(\prod_{i\in I} D_i)_{\mathcal{U}}$ of Egyptian domains. Assume further that there is a uniform finite bound on the number of distinct unit fractions needed to express the nonzero elements of D_i as a sum of such terms (as i ranges over I). This can be expressed as a first order sentence in the language of rings. For concreteness, we will show how to do this when the bound is 2 as follows:

"For all nonzero x, there exists a nonzero y such that $xy = 1$ or there exist distinct nonzero y and z such that $xyz = y + z$." It follows that the ultraproduct remains Egyptian in this case.

2.4. **Some general results.** We now present some general results and indicate further avenues of exploration. Note first that every domain D is an intersection of Egyptian overrings: simply observe that (as is well-known) $D = \bigcap_{J \in \mathrm{Max}(D)} D_J$; since each D_J is local, we can apply Corollary 1 to conclude that each D_J is Egyptian. On the other hand, the collection of Egyptian overrings of D need not be closed under arbitrary intersections: consider the polynomial ring $F[X]$, where F is a field. Then $F[X]$ is a UFD, thus is integrally closed. It follows that $F[X]$ is the intersection of its valuation overrings. Being local, each such overring is Egyptian. But by Proposition 1, $F[X]$ is not Egyptian.

A natural question which we address now is whether there are examples of domains for which every nonzero element can be represented as a sum of unit fractions, but for which some elements cannot be so expressed as sums of *distinct* unit fractions. We show that, in fact, the distinctness requirement is a superfluous assumption.

Theorem 2. *Let D be an integral domain. If a nonzero $d \in D$ can be expressed as a sum of unit fractions (in the quotient field of D), then d can be so expressed with all denominators in the expression distinct.*

Proof. Let D be a domain. We consider the cases where D has positive characteristic and characteristic zero separately.

Case 1. D has characteristic p for some prime p. We proceed by induction. Let n be a positive integer and suppose that every nonzero sum of fewer than n unit fractions can be expressed as the sum of (possibly a different number of) distinct unit fractions, and consider a nonzero sum $\frac{1}{d_1} + \cdots + \frac{1}{d_n}$, where each $d_i \in D$ is nonzero. If all d_i are distinct, we are done. So suppose without loss of generality that $n > 1$ and $\frac{1}{d_1} = \frac{1}{d_2}$. Assume first that $p = 2$. Since the sum $\frac{1}{d_1} + \cdots + \frac{1}{d_n}$ is *nonzero*, we must have $n > 2$. Noting that $\frac{1}{d_1} + \frac{1}{d_2} = 0$, we can eliminate the first two terms of the sum and then invoke the inductive hypothesis. Suppose now that $p > 2$. Then $2_D := 1_D + 1_D$ is invertible in D. Thus $\frac{1}{d_1} + \frac{1}{d_2} = \frac{2_D}{d_1} = \frac{1}{\frac{1}{2_D} d_1}$. Now apply the inductive hypothesis.

Case 2. D has characteristic 0. An arbitrary nonzero sum of unit fractions can be expressed in the form $\frac{n_1}{d_1} + \cdots + \frac{n_r}{d_r}$ for some positive integers n_i (which we naturally identify in D: since D has characteristic 0, the prime subring of D is isomorphic to $\mathbb{Z}$) and distinct d_i. Express n_1 as a sum of distinct positive unit fractions in $\mathbb{Q}$. Now multiply through by $\frac{1}{d_1}$ to obtain an Egyptian expression S_1 for $\frac{n_1}{d_1}$. If $n = 1$, we are clearly done. So suppose $n > 1$. Because D has characteristic 0, there are but finitely many positive integers k for which kd_2 is equal to a denominator of a summand in S_1. Theorem 1 implies that we can express n_2 as a sum of distinct unit fractions such that for all denominators k of this sum, kd_2 is distinct from all denominators appearing in the sum S_1. By multiplying by $\frac{1}{d_2}$, we obtain an Egyptian expression S_2 for $\frac{n_2}{d_2}$ for which the denominators are distinct from the denominators appearing in the sum S_1. Now continue this process to obtain an Egyptian expression for $\frac{n_1}{d_1} + \cdots + \frac{n_r}{d_r}$. $\square$

We obtain the following corollaries. We leave the easy proof of the first to the reader.

Corollary 4. *Let D_1 be an Egyptian domain, and suppose that D_2 is a unital domain extension of D_1 which is generated as a D_1-algebra by units. Then D_2 is Egyptian.*

As an application, consider now a commutative ring D without zero divisors and without an identity. One may still form the field K of quotients of D and embed D into K via $d \mapsto \frac{xd}{x}$, where x is a fixed nonzero element of D (we identify D with its image in K). Thus one may define the Egyptian property for D without the need for a multiplicative identity. However, we can reduce to the unital case without loss of generality, as our next corollary shows.

Corollary 5. *Let D be a commutative ring without zero divisors, and let D^1 be the unital domain obtained by adjoining an identity to D. Then D is Egyptian if and only if D^1 is Egyptian.*

Proof. Let D be as stated, and let K be the quotient field of D. Now set $D^1 := D[1_K]$, and observe that every member of D^1 can be expressed in the form $m \cdot 1_K + d$ for some $m \in \mathbb{Z}$ and $d \in D$. Now, let us assume first that D is Egyptian. As D is a domain, we see that $\mathbb{Z}1_K \cong \mathbb{F}_p$ for some prime p or $\mathbb{Z}1_K \cong \mathbb{Z}$. Either way, we may express $m \cdot 1_k$ as a sum of unit fractions (with denominators in $D[1_K]$) for any nonzero integer m. Since D is Egyptian, we can do the same for any nonzero $d \in D$. Thus every nonzero $m \cdot 1_K + d \in D[1_K]$ can be expressed as a sum of unit fractions, and so $D[1_K]$ is Egyptian by Theorem 2. Conversely, suppose that $D[1_K]$ is Egyptian, and let $d \in D$ be nonzero. Since $D[1_K]$ is Egyptian, we have $d^2 = \frac{1}{d_1} + \cdots + \frac{1}{d_k}$, where each $d_i \in D[1_K]$. But then $d = \frac{1}{dd_1} + \cdots + \frac{1}{dd_k}$ with $dd_i \in D$, and we see that D is Egyptian. □

Recall that Proposition 1 implies that $F[X]$ is not Egyptian for any field F. However, we do have the following result.

Proposition 7. *Let F be a field. Every proper overring of $F[X]$ is Egyptian.*

Proof. Let F be a field, and consider a proper overring D of $F[X]$. Since $F[X]$ is a PID, D is a quotient ring of $F[X]$. Specifically, let $S := \{f(X) \in F[X]\colon \frac{1}{f(X)} \in D\}$. Then $D = F[X]_S$ (see [7] for further reading on rings with the QR property, that is, every overring is a quotient ring). By Theorem 2, it suffices to show that every member of D can be expressed as a sum of unit fractions with denominators in D. Clearly, it suffices to show this for a subset of D which generates D as a ring. Thus it suffices to show that $\frac{1}{f(X)}$, a, and X can be so expressed for any $f(X) \in S$ and $a \in F^\times$. Toward this end, let $f(X) \in S$. Then note that since D is an overring of $F[X]$, $\frac{1}{f(X)}$ is a unit fraction with denominator in D. If $a \in F^\times$, then $a = \frac{1}{a^{-1}}$, and as above, $a^{-1} \in D$. Now, since D is a *proper* overring of $F[X]$, it follows that S contains some $g(X) \in F[X]$ of positive degree; say $g(X) = a_0 + a_k X^k + \cdots + a_n X^n$, where $a_i \in F$ and $a_k, a_n \neq 0$ (note that $k = n$ is possible). Observe that $g(X) = \frac{1}{\frac{1}{g(X)}}$ and $\frac{1}{g(X)} \in D$. It is clear that $g(X) - a_0 = a_k X^k + \cdots + a_n X^n$ is a sum of at most two unit fractions with denominators in D; call this sum S. But then $X(a_k X^{k-1} + \cdots + a_n X^{n-1}) = S$. Multiply both sides of the equation by $\frac{1}{a_k X^{k-1} + \cdots + a_n X^{n-1}}$ to obtain X as a sum of unit fractions with denominators in D, concluding the proof. □

Remark 4. The previous example shows that an intersection of a chain of Egyptian overrings need not be Egyptian. Indeed, let F be a field, and for every positive integer n, let S_n be the multiplicative subset of $F[X]$ generated by X^{2^n}. Then the collection $\mathcal{C} := \{F[X]_{S_n} : n \in \mathbb{Z}^+\}$ is a chain of proper overrings of $F[X]$; each member is Egyptian by Proposition 7. But the intersection is $F[X]$, which is not Egyptian.

Next, we make an element-wise definition of the Egyptain property, and then define several Egyptian-like subrings of an integral domain.

Definition 4. Let D be a domain. Call a nonzero $d \in D$ an ***Egyptian element*** of D provided d can be expressed as a sum of unit fractions in the quotient field of D (hence by Theorem 2, every such element has a representation as the sum of *distinct* unit fractions).

Proposition 8. *Let D be a domain. Then the following hold (all subrings are unital):*

1. *Let $E_1(D)$ be the subring of D generated by the units of D. Then $E_1(D)$ is an Egyptian subring of D.*
2. *D possesses a unique maximal Egyptian subring $E_2(D)$ (which may coincide with D).*
3. *Let $E_3(D)$ be the set of Egyptian elements of D along with 0. Then $E_3(D)$ is a subring of D.*
4. *Let S be the set of non-zero, non-Egyptian elements of D. Then S is multiplicatively closed; moreover, D_S is an Egyptian overring of D. Set $E_4(D) := D_S$.*
5. *$E_1(D) \subseteq E_2(D) \subseteq E_3(D) \subseteq E_4(D)$.*

Proof. Let D be a domain.

(1) Observe that $E_1(D) = \{u_1 + \cdots + u_n : u_i \in D^\times, i \in \mathbb{Z}^+\}$. Moreover, for any $u_1, \ldots, u_n \in D^\times$, we have $d := u_1 + \cdots + u_n = \frac{1}{u_1^{-1}} + \cdots + \frac{1}{u_n^{-1}}$, showing that $E_1(D)$ is Egyptian.

(2) Note first that the prime subring $P(D)$ of D is either a finite field or is isomorphic to $\mathbb{Z}$, and is thus Egyptian. Via Zorn's lemma, extend to a maximal Egyptian subring D^* of D. Now suppose that D_1 and D_2 are maximal (unital) Egyptian subrings of D. Then it is easy to see that D_1D_2 is also an Egyptian subring of D. Moreover, D_1D_2 contains both D_1 and D_2 as subrings. Maximality implies that $D_1 = D_2 = D_1D_2$.

(3) This follows easily from Theorem 2.

(4) Let S be the set of non-zero non-Egyptian elements of D. Note that if $x, y \in D\backslash\{0\}$ and xy is Egyptian, then by dividing through an Egyptian representation for xy by x and y, respectively, we see that x and y are Egyptian elements. This shows that S is multiplicatively closed (possibly empty). Now let $\frac{d}{s} \in D_S$ be nonzero. Suppose first that $d \in S$. Then $\frac{d}{s} = \frac{1}{\frac{s}{d}}$ gives an Egyptian representation of $\frac{d}{s}$. Now assume that $d \notin S$. Then d has an Egyptian representation with denominators in D. Now multiply through by $\frac{1}{s}$.

(5) Straightforward. □

Some comments are now in order. Let D be a domain. In a sense, the rings $E_i(D)$ defined above give a rough measure of the 'Egyption-ness' of D. On the Egyptian side of the spectrum, we see that $D = E_1(D)$ if and only if every element of D is a sum of units of D if and only if $E_1(D) = E_2(D) = E_3(D) = E_4(D)$. On the other, we may consider D to be strongly non-Egyptian if D is not a field and $E_4(D) = K$, the quotient field of D. We present examples illustrating that two of the containments can be proper.

Example 6. *The following hold:*

(1) *Let $D := \mathbb{Z}[\sqrt{-2}]$. Then $E_1(D) = \mathbb{Z}$ and $E_2(D) = D$.*
(2) *Let F be a field. Then $E_4(F[X]) = F(X)$, the field of rational functions in the variable X over F.*

Proof. We establish each claim in succession.

(1) Let D be as stated above. It is immediate from Proposition 5 that D is Egyptian, and hence $E_2(D) = D$. To show that $E_1(D) = \mathbb{Z}$, it suffices to show that ± 1 are the only units of D. It is well-known that D is Euclidean, with norm N given by $N(a + b\sqrt{-2}) := a^2 + 2b^2$. Now, $a + b\sqrt{-2}$ is a unit if and only if $N(a + b\sqrt{-2}) = 1$ if and only if $a^2 + 2b^2 = 1$ if and only if $b = 0$ and $a = \pm 1$.

(2) It follows as in the proof of Proposition 1 that any $f(X) \in F[X]$ of positive degree has no Egyptian representation. Thus for any $\frac{f(X)}{g(X)} \in F(X)$, we see that $\frac{f(X)}{g(X)} = \frac{Xf(X)}{Xg(X)} \in E_4(F[X])$. □

We conclude the subsection with an application of Proposition 8. Recall that a domain D is called a ***non-D-ring*** provided there is a non-constant polynomial $f(X) \in D[X]$ (called a *uv* (unit-valued) polynomial) such that $f(d)$ is a unit of D for all $d \in D$. These domains yield another class of examples of Egyptian domains; we refer the reader to [9] for further reading on non-D-rings (the bibliography of [9] references many earlier papers on the topic).

Example 7. *Every non-D-ring is Egyptian.*

Proof. Let D be a non-D-ring. If D is finite, then D is a field, and we are done by Proposition 1. Now assume that D is infinite. Let $f(X) := a_0 + a_1X + \cdots + a_nX^n \in D[X]$ be a non-constant polynomial that assumes only unit values on D. Because D is a domain, the nonzero polynomial $g(X) := a_1 + a_2X + \cdots + a_nX^{n-1}$ has but finitely many roots in D. Let $d \in D$ be such that $g(d) \neq 0$. We will show that d is Egyptian. Toward this end, since $f(X)$ is unit-valued, $f(0) = a_0$ is a unit of D. But also $f(d) = a_0 + a_1d + \cdots + a_nd^n := u$ is a unit of D. Therefore, $a_1d + \cdot + a_nd^n = u - a_0$. Factoring d out, $d \cdot g(d) = u - a_0$. Since $g(d) \neq 0$, we may divide through by $g(d)$ to obtain $d = \frac{u}{g(d)} - \frac{a_0}{g(d)} = \frac{1}{u^{-1}g(d)} - \frac{1}{a_0^{-1}g(d)} \in E_3(D)$. We have shown that $D \setminus E_3(D)$ is finite; since D is infinite, so is $E_3(D)$. By Theorem 2, it suffices to show that $D \setminus E_3(D) = \varnothing$. Suppose not. By Proposition 8, $E_3(D)$ is a proper, additive subgroup of $(D, +)$. Choose $x \in D \setminus E_3(D)$. Then $E_3(D) \cap (E_3(D) + x) = \varnothing$, and hence $E_3(D) + x$ is an infinite subset of $D \setminus E_3(D)$, a contradiction. □

2.5. **Directions for further research.** We close the paper with some problems for further research.

Problem 1. Classify the Egyptian subrings of the field of real numbers.

Problem 2. Study domains D for which some of the containments $E_1(D) \subseteq E_2(D) \subseteq E_3(D) \subseteq E_4(D)$ are equalities. In particular, can $E_2(D) \neq E_3(D)$ for some domain D?

Remark 5. Let D be a domain and suppose that for every $a \in E_3(D)$, there exist $b, c \in D$ for which $a = \frac{1}{b} + \frac{1}{c}$. Then $abc = b + c$, from which it follows that $b|c$ and $c|b$. Hence there is $u \in D^\times$ such that $ub = c$. From $a = \frac{1}{b}+\frac{1}{c}$, we obtain $ab = 1+\frac{b}{c}$. Thus $b = \frac{1}{a}+\frac{b}{ac} = \frac{1}{a}+\frac{b}{aub} = \frac{1}{a}+\frac{1}{au} \in E_3(D)$. Similarly, $c \in E_3(D)$. Hence $E_3(D)$ is Egyptian in this case, and therefore $E_2(D) = E_3(D)$.

Problem 3. Study the Egyptian property for PIDs, Dedekind domains, Prüfer domains, etc.

Problem 4. Study classes of Egyptian domains D for which there is a finite bound on the number of terms in an Egyptian sum. More generally, for a given nonzero element d of an arbitrary Egyptian domain, study the smallest possible representations of d as a sum of distinct unit fractions.

Problem 5. Extend the definition of 'Egyptian domain' to rings with zero divisors by considering sums of distinct unit fractions in the total quotient ring.

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