Math 4130-5130 Homework 3 Solutions

Throughout, appeal to field and vector space axioms where appropriate to justify your steps completely.

(1)[10 pts] Consider again the vector space $V = \mathcal{F}(X, F)$, where X is a nonempty set, F is a field, and where

1. $(f + g)(x) = f(x) + g(x)$, and
2. $(af)(x) = a \cdot f(x)$ for all $x \in X$ and $a \in F$.

Prove vector space axiom (3). To do this, you need to tell me what the zero vector of V is (define it), and then prove that it has the property it must have to satisfy the axiom.

Proof. Let X be a nonempty set, F be a field, and let $V = \mathcal{F}(X, F)$. Now define $z \colon X \to F$ by $z(x) = 0$ for every $x \in X$ (where 0 is from the additive identity axiom for fields). Note that $z \in V$. Now let $f \in V$ be arbitrary. We must show that $f + z = f$. Note that by definition, f and $f + z$ both have domain X. Now let $x \in X$ be arbitrary. Then $(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x)$ by field axiom (3). This proves that $f + z = f$, and so z is the zero vector of V. $\square$

(2)[10 pts] With the same setup as above, prove that vector space axiom (6) holds.

Proof. Let X be a nonempty set, F be a field, and let $V = \mathcal{F}(X, F)$. Now let $a, b \in F$ and $f \in V$ be arbitrary. We must show that $(ab)f = a(bf)$. Let $x \in X$ be arbitrary. Then note that $[(ab)f](x) = (ab)(f(x)) = a(bf(x)) = a[(bf)(x)] = [a(bf)](x)$ by definition of the scalar product and associativity of multiplication in F. As both functions have the same domain, this proves that $(ab)f = a(bf)$, as desired. $\square$

(3)[10 pts] Let V be a vector space over a field F. Prove that for all vectors $\vec{v} \in V$, if $\vec{v} + \vec{v} = \vec{v}$, then $\vec{v} = \vec{0}$.

Proof. Let V be a vector space over a field F and let $\vec{v} \in V$ be arbitrary. Assume that $\vec{v} + \vec{v} = \vec{v}$. By VS Axiom 3, the previous equation becomes $\vec{v} + \vec{v} = \vec{v} + \vec{0}$. By Cancellation, $\vec{v} = \vec{0}$, as desired. □

(4)[10 pts] Let V be a vector space over a field F, and let a be a nonzero element of F and $\vec{v} \in V$ be arbitrary. Prove that there exists a $b \in f$ such that $b(a\vec{v}) = \vec{v}$.

Proof. Let F be a field, $a \in F \backslash \{0\}$, V a vector space over F, and $\vec{v} \in V$. Since F is a field and $a \neq 0$, there exists $a^{-1} \in F$ such that $aa^{-1} = 1$ (field axiom 9). Now, $a^{-1}(a\vec{v}) = (a^{-1}a)\vec{v}$ by vector space axiom 6. Applying field axiom 6, $(a^{-1}a)\vec{v} = (aa^{-1})\vec{v}$, which becomes $1\vec{v}$. Finally, by vector space axiom 5, $1\vec{v} = \vec{v}$, and the proof is complete. □