Math 4130-5130 Homework 4 Solutions

Throughout, appeal to field and vector space axioms where appropriate to justify your steps completely.

(1)[10 pts] Let V be a vector space over a field F and suppose that W_1 and W_2 are subspaces of V. Prove that $W_1 \cap W_2$ is a subspace of V.

Proof. Let V be a vector space over a field F and suppose that W_1 and W_2 are subspaces of V. We will use the 'Subspace Theorem' to do this.

First, since W_1 and W_2 are subspaces of V, by the Subspace Theorem, $\vec{0} \in W_1$ and $\vec{0} \in W_2$. Thus $\vec{0} \in W_1 \cap W_2$. Next, let $\vec{w}_1, \vec{w}_2 \in W_1 \cap W_2$. Then $\vec{w}_1, \vec{w}_2 \in W_1$. Since (by the Subspace Theorem) W_1 is closed under addition, $\vec{w}_1 + \vec{w}_2 \in W_1$. Similarly, $\vec{w}_1 + \vec{w}_2 \in W_2$. Thus $\vec{w}_1 + \vec{w}_2 \in W_1 \cap W_2$. Lastly, suppose that $\vec{w} \in W_1 \cap W_2$ and that $a \in F$. Then $\vec{w} \in W_1$ and $\vec{w} \in W_2$. By the Subspace Theorem, $a\vec{w} \in W_1$ and $a\vec{w} \in W_2$, that is, $a\vec{w} \in W_1 \cap W_2$, and the proof is complete. □

(2)[10 pts] Let V be a vector space over a field F, and suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are elements of V. Let $W = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n : c_1, c_2, \ldots, c_n \in F\}$. Prove that W is a subspace of V which contains each of the vectors $\vec{v}_1, \ldots, \vec{v}_n$.

Proof. Let V be a vector space over a field F, let $\vec{v}_1, \ldots, \vec{v}_n \in V$, and let W be defined as above. We must check several items.

(i) $\vec{0} \in W$: note that $\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_n$ (recall from the notes that $0\vec{v}_i = \vec{0}$ for $1 \le i \le n$ and by a repeated application of vector space axiom 3, the above equation follows).

(ii) W is closed under addition: consider elements $\vec{x}, \vec{y} \in W$. Then by definition of W, we have $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ for some $c_1, \ldots, c_n \in F$ and $\vec{y} = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$ for some $d_1, \ldots, d_n \in F$. Hence $\vec{x} + \vec{y} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n + d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$. By repeated application of vector space axiom 1, $\vec{x} + \vec{y} = c_1\vec{v}_1 + d_1\vec{v}_1 + \cdots + c_n\vec{v}_n + d_n\vec{v}_n$. Now by repeated application of vector space axiom 7, $\vec{x} + \vec{y} = (c_1 + d_1)\vec{v}_1 + \cdots + (c_n + d_n)\vec{v}_n$, showing that $\vec{x} + \vec{y} \in W$.

(iii) W is closed under scalar multiplication: let $\vec{x} \in W$ and $a \in F$. Then by definition of W, $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ for some $c_1, \ldots, c_n \in F$. So $a\vec{x} = a(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = a(c_1\vec{v}_1) + \cdots + a(c_n\vec{v}_n)$ by repeated application of vector space axiom 7. Now by vector space axiom 6, $a(c_1\vec{v}_1) + \cdots + a(c_n\vec{v}_n) = (ac_1)\vec{v}_1 + \cdots + (ac_n)\vec{v}_n \in W$.

(iv) W contains each vector $\vec{v}_i$ for $1 \le i \le n$. It suffices without loss of generality to show that $\vec{v}_1 \in W$ (why?). Note that $\vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n$ by vector space axioms 5,3, and the notes (recall that $0\vec{v} = \vec{0}$ for every $\vec{v} \in F$). This proves that $\vec{v}_1 \in W$. □

(3)[5130 only][10 pts] Let V be a vector space over a field F and let $\vec{v}_1, \ldots, \vec{v}_n$ and W be as in (2). Now suppose that S is any subspace of V which contains $\vec{v}_1, \ldots, \vec{v}_n$. Prove that $W \subseteq S$. To do this, you should use induction (and understand how to apply induction here).

Proof. The proof is by induction on n. Let V be a vector space over a field F.

(i) (base case) Let $\vec{v}_1 \in V$ and let $W = \{c\vec{v}_1 : c \in F\}$. Further, let S be a subspace of V which contains $\vec{v}_1$. We must show that $W \subseteq S$. Since $\vec{v}_1 \in S$ and since S is a subspace of V, S is closed under scalar multiplication. So for any $c \in F$, we have $c\vec{v}_1 \in S$, which shows that $W \subseteq S$.

(ii) (inductive step) Let n be a positive integer, and suppose the result is true for n. Now, let $\vec{v}_1, \ldots, \vec{v}_{n+1} \in V$ and W be defined as in (2) for this collection of vectors. Suppose that S is a subspace of V which contains $\vec{v}_1, \ldots, \vec{v}_{n+1}$. We must show that $W \subseteq S$. Toward this end, S is a subspace of V containing $\vec{v}_1, \ldots, \vec{v}_n$. So by the inductive hypothesis, $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in S$ for any $c_1, \ldots, c_n \in F$. As in the base case, since $v_{n+1} \in S$ and S is closed under scalar multiplication, $c_{n+1}\vec{v}_{n+1} \in S$ for any $c_{n+1} \in F$. Finally, since S is closed under addition, $c_1\vec{v}_1 + \cdots + c_{n+1}\vec{v}_{n+1} \in S$ for any $c_1, \ldots, c_{n+1} \in F$, and this shows that $W \subseteq S$. □

(4)[10 pts] Let $F = \mathbb{R}$ and let $V = \mathbb{R}$. Find all subspaces of V (prove that they are subspaces and then prove there can't be any other).

Proof. It follows from the notes that $\{0\}$ and V are subspaces of V (you can prove this via the Subspace Theorem/definition of 'subspace', but I won't do this here). It remains to show that there are no more. So suppose that $W \neq \{0\}$ is a subspace of V. Since $0 \in W$ (Subspace Theorem), there must be some nonzero real number $r \in W$. Now let $s \in \mathbb{R}$ be arbitrary. Then observe that $s = (sr^{-1})r \in W$ since W is closed under scalar multiplication. This proves that $W = \mathbb{R}$, and so there are no more subspaces. □

(5)[5130 only][10 pts] We have seen that the set W of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Show that no member of W spans W. Note that the solution I gave today doesn't work here since the g I defined is not continuous.

Proof. Let W be the subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consisting of the continuous $f\colon \mathbb{R} \to \mathbb{R}$. Suppose by way of contradiction that $f \in W$ spans W. Note first that f cannot be a constant function: if $f(x) = r$ for some real number r and all real numbers x, then $g(x) = x^2$ (which is continuous) is not a scalar multiple of f, lest g itself be constant, which is isn't. Now consider the function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x) + 1$. Since g is the sum of two continuous function ($f(x)$ and a constant function), g is continuous. Since f spans W, there is some real number s such that $sf = g$, that is, $sf(x) = g(x)$ for all real numbers x. But this means that $sf(x) = f(x) + 1$ for all real numbers x. Doing some basic algebra, this becomes $f(x)(s-1) = 1$ for all real numbers x. Observe that $s - 1 \neq 0$ (why?), and so $f(x) = \frac{1}{s-1}$ for all real numbers x, implying that f is a constant function, a contradiction to the above. □

(6)[10 pts] Let F be a field, and consider the vector space $P(F)$ of all polynomials with coefficients in F. Prove that no finite subset of $P(F)$ spans $P(F)$.

Proof. Let F be a field and let $P(F)$ be the vector space of all polynomials with coefficients in F. Let $S \subseteq P(F)$ be finite. We will show that $\operatorname{Span}(S) \neq P(F)$.

Case 1: $S = \varnothing$. Then $\operatorname{Span}(S) = \{0\}$, and so $x \notin \operatorname{Span}(S)$.

Case 2: S is nonempty; say that $S = \{f_1(x), \ldots, f_n(x)\}$ with degrees $d_1, \ldots, d_n$, respectively. Now let $m = \max(d_1, \ldots, d_n) + 1$. Then observe that $x^m \notin \operatorname{Span}(S)$, since any linear combination of members of S is a polynomial of degree at most $\max(d_1, \ldots, d_n)$. □