MATH 4130-5130 Autumn 2023 Lecture 4

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Fields

Definition

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9. For every $a \in F$: if $a \neq 0$, then there is $b \in F$ such that $a \cdot b = 1$. (multiplicative inverse axiom)
10. $a \cdot (b+c) = a \cdot b + a \cdot c$ for every $a, b, c \in F$. (distributivity axiom)

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Proof.

(direct proof) Let F be a field and let $a, b, c \in F$ be arbitrary.

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We can now easily show that we can "cancel" on the left as well as the right.

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(direct proof) Let F be a field and let $a, b, c \in F$ be arbitrary. Assume that $a + b = a + c$. We will show that $b = c$. By axiom 2, the previous equation can be changed to $b + a = c + a$.

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The previous example and its corollary are often called the *additive cancellation laws*. We can use these laws to show that the element 0 postulated in axiom 3 is unique.

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Suppose that F is a field. Prove that $0 \in F$ is *unique*[1], that is, show that there is no element $0' \in F$ different from 0 with the property that $a + 0' = a$ for every $a \in F$.

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Remark

To prove uniqueness, one can also assume that if x and y are particular objects with a certain property, then $x = y$.

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The associative property of addition renders parentheses in expressions between elements of a field where the only operation is $+$ redundant.

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Suppose that F is a field. For any $a \in F$, the unique $b \in F$ such that $a + b = 0$ is commonly denoted by $-a$.

Remark

The associative property of addition renders parentheses in expressions between elements of a field where the only operation is $+$ redundant. This can be proved rigorously but I am not going to torture you with the proof. For example, $(a + b) + (c + d) = a + (b + (c + d)) = (a + (b + c)) + d$, etc. and so we can simply write "$a + b + c + d$" with no parentheses without ambiguity.

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