MATH 4130-5130 Autumn 2022 Lecture 5

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Vector Spaces

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Let $F = \mathbb{R}$, the field of real numbers and $V = \mathbb{R}^n = \{(r_1, r_2, \ldots, r_n)\colon \text{ each } r_i \in \mathbb{R}\}$.

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The set $\mathcal{F}(\mathbb{N}, \mathbb{R})$ then denotes all functions $f\colon \mathbb{N} \to \mathbb{R}$, that is, all functions with domain the set $\mathbb{N}$ of natural numbers and codomain the set $\mathbb{R}$ of real numbers.

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Let $X = \{2\}$ (that is, the set whose only member is the real number 2). Define $f: X \to \mathbb{Q}$ by $f(x) = 2x$ and $g: X \to \mathbb{Q}$ by $g(x) = x^2$. Then note that by definition, f and g have the same domain, namely X. Also, observe that the only member of D is 2 and we have $f(2) = 2 \cdot 2 = 4 = 2^2 = g(2)$, and thus $f = g$.

Vector Spaces

Definition

Suppose that f and g are functions. Then $f = g$ exactly when the domain of f and the domain of g are the same AND $f(x) = g(x)$ for EVERY x in the common domain.

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Let V be a vector space over a field F. Then the following hold:

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Vector Spaces

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