MATH 4130-5130 Autumn 2023 Lecture 6

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Proof.

Let F be a field and let $P_n(F)$ be defined as above. We check the three conditions:

Subspaces

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Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and let W be the subset of V consisting of the functions which are continuous on $\mathbb{R}$.

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