MATH 4130-5130 Lecture 7

Greg Oman

University of Colorado
Colorado Springs

Linear Combinations and Span

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Before proceeding, let us do an example (we will need to use this example shortly) of showing that a certain subset of a vector space is a subspace.

Example

Let V be a vector space over a field F. Then $\{\vec{0}\}$ is a subspace of V, where $\vec{0}$ is the zero vector of V.

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The members of Span(S) are precisely the vectors of the form $a(1,0,0) + b(0,0,1) + c(0,0,1)$, where a, b, c are any real numbers.

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Suppose by way of contradiction that some $f \in V$ spans V. Let's show that $f(r) \neq 0$ for any real r. Suppose by way of contradiction that there is some real r such that $f(r) = 0$. Now let $g(x) = 1$ for every real number x. Since f spans V, there is some real number a such that $af = g$. Hence $(af)(r) = g(r)$, and so $af(r) = g(r)$. But then $0 = g(r) = 1$, a contradiction. Thus $f(x) \neq 0$ for every real number x. Now let $g(x) = 0$ for $x \neq 0$ and $g(0) = 1$. Then there is some real number a such that $af = g$.Hence $(af)(1) = g(1) = 0$, so $af(1) = 0$. Since $f(1) \neq 0$, $a = 0$.

Linear Combinations and Span

Example

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Proof.

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