Math 4130-5130 Homework 2 Solutions

(1)[2 pts each] Determine if the following are operations on the given set. If so, you do not need to justify your work. If not, tell me which part(s) of the definition fails (see notes).

(a) Let $S = \mathbb{N}$ (I defined this set in the notes) and let $*$ be defined by $a * b = a - b$.

Solution. This is not an operation as, for example, $1 * 2 \notin S$. □

(b) Let $S = \{1, -1\}$ and let $*$ be defined by $a * b = \frac{a}{b}$.

Solution. This is an operation. □

(c) Let $S = \mathbb{Z}$ and let $*$ be defined by $a * b = \frac{a}{b}$.

Solution. This is not an operation as there are integers a and b for which $a * b \notin \mathbb{Z}$ AND the operation is not defined when $b = 0$. □

(d) Let S be the set consisting of all 2×3 and 3×2 matrices with real coefficients, and let $*$ be defined by $A * B = A + B$.

Solution. The sum of a 2×3 and a 3×2 matrix is not defined, so $*$ is not defined for every pair of elements of S. □

(e) Let S be the same set as in (d), but define $A * B$ by $A * B = AB$ (this is intended to denote matrix multiplication, which you should have learned in a first course in linear algebra).

Proof. This is also not an operation for similar reasons: the product of two 2×3 matrices is undefined, and the product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix, which is not in S. □

(2)[2 pts each] Determine if the following are scalar multiplications on the given set. If so, you do not need to justify your work. If not, tell me which part(s) of the definition fails.

(a) Let $S = \mathbb{Z}$ and define $\cdot$ by $r \cdot m = rm$ (the usual multiplication of real numbers).

Solution. This is not a scalar multiplication since, for example, $\sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Z}$. So closure fails. □

(b) Let $S = \{0\}$ and define $\cdot$ by $r \cdot m = rm$ for $m \in S$.

Solution. This is a scalar multiplication on S. □

(c) Let $S = \{1, 2, 3\}$ and define $\cdot$ by $r \cdot m = 2$ for $m \in S$.

Solution. This is a scalar multiplication on S. □

(d) Let $S = \mathbb{C}$ and define $\cdot$ by $r \cdot (a + bi) = a$.

Solution. This is a scalar multiplication on S. □

(3) Prove the following using the axioms for a field (you may also use any results I proved in the notes if they are useful). Please cite the axioms you appeal to as I did in the notes. Throughout, F is an arbitrary field.

(a) [not graded] Prove that 1 is unique; that is, prove that there is no element $1' \in F$ different from 1 with the property that $1' \cdot a = a$ for all $a \in F$.

Proof. Suppose by way of contradiction that there exists a field F and distinct elements $1, 1' \in F$ such that $1 \cdot a = a$ for all $a \in F$ and $1' \cdot a = a$ for all $a \in F$. Then in particular, we have $1 \cdot 1' = 1'$ and $1' \cdot 1 = 1$. By commutativity of multiplication, $1' \cdot 1 = 1 \cdot 1'$. Thus $1 \cdot 1' = 1'$ and $1 \cdot 1' = 1$. This proves that $1 = 1'$, a contradiction. □

(b)[not graded] Prove that for any $a \in F$, $0 \cdot a = 0$. Using this result, prove that $a \cdot 0 = 0$ also (using the first result; this should only be a line or so: you just need to appeal to one of the multiplicative axioms). [Hint for the first part: $0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$ (justify this; there's a bit more to do to finish the proof).]

Proof. Let $a \in F$ be arbitrary, where F is a field. Then note that $a \cdot 0 = a \cdot (0 + 0)$ by the additive identity axiom. So we have $a \cdot 0 = a \cdot (0+0)$. By the distributive axiom, $a \cdot (0+0) = a \cdot 0 + a \cdot 0$. Thus $a \cdot 0 = a \cdot 0 + a \cdot 0$. Now by the additive identity axiom again, $a \cdot 0 + 0 = a \cdot 0 + a \cdot 0$. By additive cancellation, $0 = a \cdot 0$, as desired. □

(c)[not graded] Prove that for any $a, b, c \in F$, we have $(a + b)c = ac + bc = ca + cb$.

Proof. Let F be a field and let $a, b, c \in F$ be arbitrary. By commutativity of multiplication, $(a + b)c = c(a + b)$; by the distributive axiom, $c(a + b) = ca + cb$. Finally, by commutativity of multiplication again, $ca + cb = ac + bc$. This proves that all three quantities are equal. □

(d)[10 pts] Prove that for any nonzero $a, b \in F$, we have $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$.

Proof. Let F be a field, and let $a, b \in F$ be nonzero. By the multiplicative inverse axiom, $a \cdot \frac{1}{a} = b \cdot \frac{1}{b} = 1$. Thus $ab\frac{1}{a}\frac{1}{b} = a\frac{1}{a}b\frac{1}{b}$ by commutativity of multiplication. From the above, $a\frac{1}{a}b\frac{1}{b} = 1 \cdot 1 = 1$ by the multiplicative identity axiom. So we have shown that $ab(\frac{1}{a} \cdot \frac{1}{b}) = 1$. Finally, note that $ab \neq 0$: if $ab = 0$, then we can multiply both sides by $\frac{1}{b}$ to get $a = 0 \cdot \frac{1}{b} = 0$ by the multiplicative inverse/identity axioms and 3(b) above. But this contradicts that a is nonzero. Finally, invoking the uniqueness of inverses, it follows that $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$, as desired. □