## Math 3410 Homework 1 Solutions

(1)[3 pts each] Let $P(x, y)$ be "$x$ is friends with $y$" and let $Q(a, b)$ be "$a$ and $b$ are roommates." Let the domain for all variables be the collection of all UCCS students. Translate the following into English (use proper grammar and punctuation).

(a) $\exists x \forall y P(x, y)$

*Solution.* For every UCCS student $x$, there exists a UCCS student $y$ such that $x$ is friends with $y$. □

(b) $\forall x \forall y (P(x, y) \wedge Q(x, y))$

*Solution.* For all UCCS students $x$ and $y$, $x$ is friends with $y$ and $x$ and $y$ are roommates. □

(c) $\exists x \forall y \exists z (P(x, y) \rightarrow (\neg Q(z, x)))$

*Solution.* There exists a UCCS student $x$ such that for every UCCS student $y$, there exists a UCCS student $z$ such that if $x$ is friends with $y$, then $z$ and $x$ are not roommates. □

(d) $\forall x \forall y ((\neg P(x, y)) \leftrightarrow (\neg Q(x, y)))$

*Solution.* For all UCCS students $x$ and $y$, $x$ is friends with $y$ if and only if $x$ and $y$ are not roommates. □

(e) $(\forall x P(John, x) \rightarrow \exists y Q(John, y))$

*Solution.* If for all UCCS students $x$, John is friends with $x$, then there exists a UCCS student $y$ such that John and $y$ are roommates. □

(2)[4 pts each] Let $P(x, y)$ be "$y = x^2$" and let $Q(x, y, z)$ be "$xy = z$". Answer the following questions.

(a) Find a common domain for $x$ and $y$ (meaning the domain is the same for both variables) for which $\exists x \exists y P(x, y)$ is true. Then find a common domain making $\exists x \exists y P(x, y)$ false (your domains should be collections of numbers).

*Solution.* True domain: $\mathbb{R}$; false domain: $(-\infty, 0)$. □

(b) Same directions as above, but for the formula $\exists x \exists y \forall z Q(x, y, z)$ (now you need a common domain for $x$, $y$, and $z$, of course)

*Solution.* True domain: $\{0\}$; false domain: $\mathbb{R}$. □

(c) Same directions as above, but for the formula $\forall x \exists y \exists z Q(x, y, z)$.

*Solution.* True domain: $\mathbb{R}$; false domain: $\{2\}$. □

(3)[3 pts each] Let $F(x, y)$ be "$x$ is friends with $y$" and let $S(x)$ be "$x$ is a student". Translate the following into formulas, where the domain for the variables is the collection of all people.

(a) Mary is friends with everyone.

*Solution.* $\forall x F(Mary, x)$. □

(b) Mary is friends with some student.

*Solution.* $\exists x(S(x) \wedge F(Mary, x)$. □

(c) There is exactly one student.

*Solution.* $\exists x(S(x) \wedge \forall y(S(y) \rightarrow y = x))$. □

(d) John is friends with everyone that Steve is friends with.

*Solution.* $\forall x(F(Steve, x) \rightarrow F(John, x))$. □

(e) No student has a friend.

*Solution.* $\forall x(S(x) \rightarrow (\neg \exists y F(x, y)))$. □

(4)[10 pts] Give a direct proof that the product of two odd integers is odd. Preformalize first.

Preformalization: $\forall x \forall x((x \text{ is odd} \wedge y \text{ is odd}) \rightarrow xy \text{ is odd})$

*Proof.* Let $x$ and $y$ be arbitrary integers. Assume that $x$ and $y$ are odd. We must show that $xy$ is odd. Since $x$ and $y$ are odd, $x = 2m + 1$ and $y = 2n + 1$ for some integers $m$ and $n$. Thus $xy = (2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$. Since $2, m$, and $n$ are integers, so is $2mn + m + n$. Thus $xy$ is odd. □

(5)[10 pts] Prove by contraposition that for all integers $x$: if $x$ is even, then $x - 1$ is odd. Preformalize first.

Preformalization: $\forall x(x \text{ is even} \rightarrow x - 1 \text{ is odd})$

*Proof.* Let $x$ be an arbitrary integer. Assume that $x - 1$ is not odd. We will show that $x$ is not even. Since $x - 1$ is not odd, $x - 1$ is even. Thus $x - 1 = 2y$ for some integer $y$. Hence $x = 2y + 1$, and so $x$ is odd. This implies that $x$ is not even. □

(6)[10 pts] Prove by contradiction that if $x$ is a nonzero rational number and $y$ is irrational, then $xy$ is irrational. Preformalize the statement to be proved AND the negation.

Preformalization: $\forall x \forall y((x$ is a nonzero rational number $\wedge$ $y$ is irrational$) \rightarrow xy$ is irrational$)$.
Negation: $\exists x \exists y((x$ is a nonzero rational number $\wedge$ $y$ is irrational$) \wedge$ $xy$ is rational$)$.

*Proof.* Assume by way of contradiction that there exists real numbers $x$ and $y$ such that $x$ is nonzero and rational, $y$ is irrational, and $xy$ is rational. Since $x$ is a nonzero rational number, $x = \frac{a}{b}$ for some nonzero integers $a$ and $b$. Since $xy$ is rational, $xy = \frac{c}{d}$ for some integers $c$ and $d$ with $d$ nonzero. Now substitute $\frac{a}{b}$ for $x$ in $xy = \frac{c}{d}$ to get $\frac{a}{b}y = \frac{c}{d}$. Solve for $y$ to obtain $y = \frac{cb}{ad}$. Since $b$ and $c$ are integers, so is $cb$; since $a$ and $d$ are nonzero integers, so is $ad$. Thus $y$ is rational, contradicting that $y$ is irrational. □

(7)[10 pts] Prove by induction that every positive integer is either even or odd (there is actually something to prove in the base case here; it's not difficult, but please do it).

*Proof.* We will prove that every positive integer is either even or odd by induction.

(i) (base case) We must show that 1 is either even or odd. Since $1 = 2 \cdot 0 + 1$, 1 is odd.

(ii) (inductive step) Let $n$ be an arbitrary positive integer, and assume that $n$ is either even or odd. We must show that $n + 1$ is either even or odd.

Case 1: $n$ is even. Then $n = 2x$ for some integer $x$. Hence $n + 1 = 2x + 1$, so $n + 1$ is odd. Hence $n + 1$ is either even or odd.

Case 2: $n$ is odd. Then $n = 2y + 1$ for some integer $y$. Thus $n + 1 = 2y + 2 = 2(y + 1)$. Because $y$ and 1 are integers, so is $y + 1$. Hence $n + 1$ is odd. Thus $n + 1$ is either even or odd.

The proof is now complete by the Principle of Mathematical Induction. □