EXTENDING SUBGROUP AND SUBRING ISOMORPHISMS TO AUTOMORPHISMS

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Abstract. Let G be an abelian group. Then G is called a T_1' *group* provided that for every pair of isomorphic subgroups H and K of G and every isomorphism $\varphi\colon H \to K$, φ can be extended to an automorphism of G. Let us call G a *weakly* T_1' *group* if for any elements a and b of G of the same order, there is an automorphism $\varphi\colon G \to G$ such that $\varphi(a) = b$. In this note, we determine both classes of groups, correcting a result in the literature. We then study several analogues for rings.

1. Introduction

A common theme in mathematics is that of extending a mathematical object to a larger one, often with stronger properties. Indeed, examples are ubiquitous: embedding a field into an algebraically closed field, a module into an injective module, a filter into an ultrafilter, or extending a linearly independent set to a basis in a vector space.

Often, these embedings give deeper insight into the structure of the original object. For example, the fact that the algebraic closure of the field $\mathbb{R}$ of real numbers is the field $\mathbb{C}$ of complex numbers allows one to quickly prove that the irreducible polynomials over $\mathbb{R}$ are the linear polynomials and the quadratic polynomials with no real roots. As another example, the existence of a basis for an arbitrary vector space enables one to prove that $\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ as additive abelian groups. We begin this note by addressing the following two questions, which are more restrictive variations on this theme (the terminology to follow can be found, for example, in [3]).

Question 1. For which abelian groups G can every isomorphism between two subgroups of G be extended to an automorphism of G, that is, which abelian groups are $\mathbf{T_1'}$?

Question 2. For which abelian groups G is it the case that if $a, b \in G$ have the same order, then there is an automorphism of G mapping a to b, that is, which abelian groups are ***weakly*** $\mathbf{T_1'}$?

Question 1 was addressed in [3]. However, the main theorem claiming to classify the T_1' abelian groups (Theorem 1.1) is somewhat flawed. As this paper has been cited several times in the literature (see, for instance, [9], [10], and [11]), we have decided to submit a correction, as well as some new results, including extensions to rings.

Before stating the first theorem of this note, we recall some terminology. An abelian group G is *divisible* if for every $g \in G$ and positive integer n, there is $h \in G$ for which $nh = g$. For a prime p, the *quasi-cyclic group of type* p^∞, denoted $\mathbb{Z}/p^\infty\mathbb{Z}$, is the direct limit of the cyclic

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groups $\mathbb{Z}/p^n\mathbb{Z}$. More concretely, $\mathbb{Z}/p^\infty\mathbb{Z} \cong \{\frac{a}{p^n} : a \in \mathbb{Z}, n \in \mathbb{Z}^+\} \pmod{\mathbb{Z}}$. A group G is called *cocyclic* provided $G \cong \mathbb{Z}/p^i\mathbb{Z}$ for some i with $0 \leq i \leq \infty$. Finally, G is *homococyclic* provided G is isomorphic to a direct sum of mutually isomorphic cocyclic groups; if $0 \leq i < \infty$, then G is called *homocyclic*. We are now ready to state the primary group-theoretic result of this article.

Theorem 1. *Let G be an abelian group. Consider the following conditions:*

(1) *G is T_1'.*
(2) *G is weakly T_1'.*
(3) *$G \cong D \oplus \bigoplus_{p \text{ prime}} G_p$, where D is a divisible torsion-free abelian group and each G_p is a homococyclic p-group.*

Then (1) $\implies$ (2) $\implies$ (3). *Further,* (3) $\implies$ (1) *if and only if D and each G_p have finite rank. Finally,* (3) $\implies$ (2).

After proving Theorem 1, we establish several corollaries. We then conclude the paper by proving some related results for rings. Throughout, we will often not give references for standard results in graduate algebra. Instead, we point the reader to the standard texts [2], [4], and [7].

2. Proof of Theorem 1

To begin, recall that for an abelian group G and prime p, the *p-component* of G is the subgroup of G consisting of all elements of G whose orders are powers (possibly the zero power) of p. Theorem 1.1 of [3] claims that G is T_1' if and only if G is torsion and all p-components are homococyclic of finite rank, or G is divisible with finite torsion-free rank and with all p-ranks finite. However, Theorem 1 shows that there are T_1' abelian groups which are neither torsion nor divisible. Before proving Theorem 1, we establish several lemmas. In what follows, all direct sums are external unless stated otherwise.

Lemma 1. *Consider a direct sum $G := \bigoplus_{i<\kappa} G_i$ of abelian groups, where κ is a cardinal and where for every finite set $J \subseteq \kappa$, $\bigoplus_{j\in J} G_j$ is weakly T_1'. Then G is weakly T_1'.*

Proof. Assume G is as stated above. Now consider elements $\mathbf{x}, \mathbf{y} \in G$ of the same order. Without loss of generality, we may assume that $\mathbf{x} = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ and that $\mathbf{y} = (y_0, y_1, \ldots, y_n, 0, 0, \ldots)$ for some $n \in \omega$. It follows from our assumption that there is an automorphism $\varphi\colon \bigoplus_{0\leq i\leq n} G_i$ for which $\varphi(x_0, \ldots, x_n) = (y_0, \ldots, y_n)$. Now extend φ to all of G as follows: $\overline{\varphi}(g_0, \ldots, g_n, \ldots) := (\varphi(g_0, \ldots, g_n), g_{n+1}, g_{n+2}, \ldots)$. One checks easily that $\overline{\varphi}(\mathbf{x}) = \mathbf{y}$ and that $\overline{\varphi}$ is an automorphism of G, as desired. □

Lemma 2. *Suppose that G is a weakly T_1' torsion abelian group. Then $G \cong \bigoplus_{p \text{ prime}} G_p$, where each G_p is a homococyclic p-group.*

Proof. It is well-known (and easy to prove) that $G \cong \bigoplus_{p \text{ prime}} G_p$, where each G_p is the p-component of G. Now let p be a prime. We will show that G_p is homococyclic. Toward this end, we consider two cases.

Case 1: G_p has elements of order p^n for arbitrarily large positive integers n. Then G has elements of order p^n for every non-negative integer n. Let $g \in G_p$ be arbitrary; say g has order p^i. Now let $h \in G_p$ have order p^{i+1}. Then there is an automorphism φ of G such that $\varphi(ph) = g$.

But then $p\varphi(h) = g$. This proves that G_p is p-divisible (it is clear from order considerations that $\varphi(h) \in G_p$). Since G_p is a p-group, G_p is q-divisible for every prime $q \neq p$. Hence G_p is p'-divisible for all primes p', which implies that G_p is divisible. By the Structure Theorem for Divisible Abelian Groups, it follows that G_p is a direct sum of copies of $\mathbb{Z}/p^\infty\mathbb{Z}$, proving that G_p is homococyclic.

Case 2: there is a bound on the orders of G_p. Then G_p is a direct sum of cyclic groups each of order a power of p; without loss of generality, write $G_p = \bigoplus_{i<\kappa} \mathbb{Z}/p^{k_i}\mathbb{Z}$, where κ is a cardinal, each k_i is a positive integer, and the direct sum is internal. It now suffices to prove that for any $i, j < \kappa$, we have $k_i = k_j$. Suppose by way of contradiction that for some $i, j < \kappa$, we have $k_i < k_j$. Let $a_i \in \mathbb{Z}/p^{k_i}\mathbb{Z}$ have order p and $b_j \in \mathbb{Z}/p^{k_j}\mathbb{Z}$ have order p^{k_j}. Then note that $p^{k_j-1}b_j$ has order p. Hence there is an automorphism φ of G such that $\varphi(p^{k_j-1}b_j) = a_i$. Now, $\varphi(b_j) = \sum_{c<\kappa} x_c$, where each $x_c \in \mathbb{Z}/p^{k_c}\mathbb{Z}$ and almost all x_c are zero. Observe that x_i has order at most p^{k_i}. As $k_i < k_j$, it follows that $p^{k_j-1}x_i = 0$. But then $\varphi(p^{k_j-1}b_j) = \sum_{c<\kappa} p^{k_j-1}x_c \neq a_i$ (since the ith component of the sum is 0), and the proof is complete. □

Lemma 3 ([3], Proposition 2.10)**.** *Suppose that G is a divisible abelian group such that the p-rank of G is finite for every prime p as is the torsion-free rank of G. Then G is T_1'.*

It should be mentioned that though the main result of [3] is flawed, Proposition 2.10 is correct. We are now in position to prove Theorem 1.

Proof of Theorem 1. Let G be an abelian group.

(1) $\implies$ (2). Assume that G is T_1'. We will show that G is weakly T_1'. So suppose that $a, b \in G$ have the same order. There is an isomorphism $\varphi\colon \langle a\rangle \to \langle b\rangle$ with $\varphi(a) = b$ (the map $ma \mapsto mb$ is easily seen to be a well-defined isomorphism, where m is an arbitrary integer). Because G is T_1', we may extend φ to an automorphism of G, concluding the proof.

(2) $\implies$ (3). Assume that G is weakly T_1'. We will prove that $G \cong D \oplus \bigoplus_{p \text{ prime}} G_p$ for some divisible torsion-free abelian group D and where each G_p is a homococyclic p-group. Let $TF(G)$ be the subgroup of G generated by 0 and the elements (if any) of infinite order (note that $TF(G)$ may contain nontrivial torsion elements). It is easy to see that $TF(G) = \{g_1 + \cdots + g_n\colon$ each g_i has infinite order$\} \cup \{0\}$. We claim that $TF(G)$ is divisible. So let $g \in TF(G)$ be arbitrary and let n be a positive integer. We must find $h \in TF(G)$ such that $nh = g$. If $g = 0$, this is patent, so suppose that $g = g_1 + \cdots + g_k$ for some $g_1, \ldots, g_k \in G$ of infinite order. Let $1 \leq i \leq k$ be arbitrary. Then g_i and ng_i both have order ∞. Hence there is an automorphism $\varphi_i\colon G \to G$ such that $\varphi_i(ng_i) = g_i$. But then $n\varphi_i(g_i) = g_i$. As φ_i is an isomorphism and since g_i has infinite order, so does $\varphi_i(g_i)$; hence $\varphi_i(g_i) \in TF(G)$. It follows that $g = g_1 + \cdots + g_k = n\varphi_1(g_1) + \cdots + n\varphi_k(g_k) = n(\varphi_1(g_1) + \cdots + \varphi_k(g_k))$. As each $\varphi_i(g_i) \in TF(G)$, also $\varphi_1(g_1) + \cdots + \varphi_k(g_k) \in TF(G)$. This proves that $TF(G)$ is divisible. But then $TF(G)$ is an internal direct summand of G; say $TF(G) \oplus H = G$. Clearly H is torsion. By the Structure Theorem for Divisible Abelian Groups, $TF(G) = D \oplus H'$ for some torsion-free divisible subgroup D (which may be trivial) of $TF(G)$ and divisible torsion subgroup H' of $TF(G)$. So now $G = D \oplus H' \oplus H$. But then for any automorphism β of G, we have $\beta(D) = D$, as D is the unique maximal torsion-free divisible subgroup of G. Thus $\beta(H' \oplus H) = H' \oplus H$. It follows that $H' \oplus H$ is also weakly T_1'. Invoking Lemma 2 concludes the proof of this implication.

Now suppose that $G \cong D \oplus \bigoplus_{p \text{ prime}} G_p$ for some divisible torsion-free abelian group D and where each G_p is a homococyclic p-group and that G is T_1'. We must show that D and each G_p have finite rank. Suppose not. Let's say that D has infinite rank (the same argument applies if some G_p has infinite rank). Then D is isomorphic to a proper subgroup of itself. Via the identity map on $\bigoplus_{p \text{ prime}} G_p$, we may extend this to an isomorphism of G with a proper subgroup of itself. But then this map cannot be extended to an automorphism of G, a contradiction.

Conversely, suppose that D and each G_p have finite rank, viewing the direct sum decomposition of G above internally. We will show that G is T_1'. Now, $G \subseteq E(G)$, where $E(G)$ is the injective hull of G, which is divisible of the same torsion-free rank and p-rank of G, for every prime p. By Lemma 3, $E(G)$ is T_1'. Now suppose that H and K are subgroups of G, and that $\varphi\colon H \to K$ is an isomorphism. We must show that we may extend φ to an automorphism of G. Now, H and K are subgroups of $E(G)$. Because $E(G)$ is T_1, there is an extension of φ to an automorphism $\overline{\varphi}$ of $E(G)$. Now, if $\overline{\varphi}(G) = G$, then $\overline{\varphi}|G$ is an automorphism of G mapping H onto K, as desired. Hence it suffices to prove that if β is an automorphism of $E(G)$, then $\beta(G) = G$ (note that this is not true for all abelian groups; if $G = \mathbb{Z}$, then the result fails, for example). Begin by noting that D is the unique maximal divisible torsion-free subgroup of $E(G)$. Hence $\beta(D)$ is the unique maximal torsion-free divisible subgroup of $\beta(E(G)) = E(G)$, whence $\beta(D) = D$. Now fix a prime q. If G_q is divisible, then as above, it follows that $\beta(G_q) = G_q$. So suppose that G_q is homococyclic but non-divisible. Then G_q is a finite direct sum of cyclic groups all of the same cardinality q^k. But then by The Structure Theorem for Divisible Abelian Groups, the set of elements of $E(G_q)$ of order at most q^k is precisely G_q. It follows that $\beta(G_q) = G_q$ in this case as well. We now deduce that $\beta(G) = G$, as desired.

(3) $\implies$ (2). By Lemma 1, it suffices to consider G in (3) of finite rank. By what we just proved above, G is T_1'. As (1) $\implies$ (2), G is weakly T_1'. The proof is now complete. □

3. Some Consequences

Observe that Theorem 1 completely characterizes the weakly T_1' abelian groups and corrects Theorem 1.1 of [3], which classifies the T_1' abelian groups. Now, let $\mathcal{F}$ be a family of T_1' abelian groups. Let us say that $\mathcal{F}$ is *homogeneous* provided that for any $G, K \in \mathcal{F}$ and any prime number p: if G_p and K_p are nonzero, then $\exp(G_p) = \exp(K_p)$, where G_p, K_p are the p-components of G and K, respectively (recall that the *exponent* of an abelian group G (written additively) is the least positive integer n for which $nG = \{0\}$, if such exists, and ∞ otherwise).

Corollary 1. *The weakly T_1' abelian groups are precisely direct sums of the members of $\mathcal{F}$, where $\mathcal{F}$ ranges over the homogeneous families of T_1' abelian groups.*

Proof. Theorem 1 shows immediately that for any such homogeneous family $\mathcal{F}$ of T_1' abelian groups, the direct sum of the members of $\mathcal{F}$ is weakly T_1'. Conversely, by Theorem 1, every weakly T_1' abelian group G can be written as a direct sum of subgroups of finite torsion-free rank and finite p-rank for every prime p, with these subgroups forming a homogeneous family of T_1' abelian groups. □

As an application, we present a new proof of the following elementary number-theoretic fact.

Corollary 2. *Let $n \in \mathbb{Z}^+$ and let a and b be integers. Then the congruence $ax \equiv b \pmod{n}$ has a solution which is relatively prime to n if and only if $\gcd(a, n) = \gcd(b, n)$.*

Proof. Clearly, we may assume that $n > 1$. Suppose first that $ax \equiv b \pmod{n}$, where x is an integer relatively prime to n. Then in $\mathbb{Z}/n\mathbb{Z}$, we have $|\overline{a}| = |\overline{ax}| = |\overline{b}|$ (where $|\ |$ denotes order). Hence $|\overline{a}| = |\overline{b}|$, from which it follows (as is well-known) that $\gcd(a, n) = \gcd(b, n)$.

Now suppose that $\gcd(a, n) = \gcd(b, n)$. Then $\overline{a}$ and $\overline{b}$ have the same order in $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{k_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{k_r}\mathbb{Z}$, where $p_1^{k_1} \cdots p_r^{k_r}$ is the prime factorization of n. It is immediate from Theorem 1 that $\mathbb{Z}/p_1^{k_1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{k_r}\mathbb{Z}$ is T_1', and thus so is $\mathbb{Z}/n\mathbb{Z}$. Because $\gcd(a, n) = \gcd(b, n)$, $|\overline{a}| = |\overline{b}|$ in $\mathbb{Z}/n\mathbb{Z}$. There is an isomorphism $\varphi\colon \langle\overline{a}\rangle \to \langle\overline{b}\rangle$ for which $\varphi(\overline{a}) = \overline{b}$, which thus may be extended to an automorphism $\overline{\varphi}$ of $\mathbb{Z}/n\mathbb{Z}$. Since $\overline{1}$ generates $\mathbb{Z}/n\mathbb{Z}$, $\overline{\varphi}(\overline{1}) := \overline{x}$ is also a generator, hence x is relatively prime to n. But now $\overline{b} = \overline{\varphi}(\overline{a}) = a\varphi(\overline{1}) = \overline{ax}$ in $\mathbb{Z}/n\mathbb{Z}$, that is, $ax \equiv b \pmod{n}$. □

Remark 1. It is not true that if $\gcd(a, n) = \gcd(b, n)$ and if x is *any* integer satisfying $ax \equiv b \pmod{n}$, then x must be relatively prime to n. As a simple example, $\gcd(2, 6) = \gcd(4, 6)$, and $2 \cdot 2 \equiv 4 \pmod{6}$, but 2 is not relatively prime to 6.

Recasting the Corollary 2 in purely algebraic terms, we obtain the following:

Corollary 3 (Corollary 2, reformulated)**.** *Let $n \in \mathbb{Z}^+$, and let a and b be integers. Then in the ring $\mathbb{Z}/n\mathbb{Z}$, there is a unit $\overline{u} \in \mathbb{Z}/n\mathbb{Z}$ such that $\overline{a} \cdot \overline{u} = \overline{b}$ if and only if $|\overline{a}| = |\overline{b}|$ in the abelian group $\mathbb{Z}/n\mathbb{Z}$.*

4. Extending to rings

4.1. **Extending Corollary 3.** Next, we consider the following natural question prompted by Corollary 3.

Question 3. Can the commutative rings R with identity $1 \neq 0$ with the property that for any $a, b \in R$ of the same additive order, there is a unit $u \in R$ such that $au = b$ be classified?

Before proceeding, recall from the Introduction that a group G is homocyclic provided G is a direct sum of mutually isomorphic cyclic groups of prime power order. Let us call a commutative ring R with identity (**henceforth, all rings in this subsection will be commutative with identity**) *order-associate* if for any $a, b \in R$ of the same additive order, there is a unit $u \in R$ for which $au = b$. We now solve Question 3.

Theorem 2. *Let R be a ring with $1 \neq 0$. Then R is order-associate if and only if R is a field of characteristic 0, or $R \cong R_1 \times \cdots \times R_n$ for some rings $R_1, \ldots, R_n$ such that*

1. *there are distinct primes $p_1, \ldots, p_n$ such that $(R_i, +)$ is a nontrivial homocyclic p_i-group for $1 \leq i \leq n$; say the exponent of $(R_i, +)$ is $p_i^{k_i}$, and*
2. *for $1 \leq i \leq n$, the ideals of R_i are precisely $I_j := \{x \in R_i\colon p^j x = 0\}$, where $0 \leq j \leq k_i$.*

Proof. Let R be a nontrivial ring. Suppose first that R is a field of characteristic 0, and let $a, b \in R$ have the same additive order. If a or b has order 1, then $a = b = 0$, and $a \cdot 1 = b$. Now suppose that a and b are nontrivial. Then choose $u := a^{-1}b$. We have shown that R is order-associate in this case.

Next, suppose that $(R, +)$ is a homocyclic p-group for some prime p of exponent p^k, $k > 0$. Further, suppose that the ideals of R are precisely the ideals $I_j := \{x \in R\colon p^j x = 0\}$, where $0 \leq j \leq k$. We will show that R is order-associate in this case. Observe trivially that for

$0 \leq i \leq j \leq k$, we have $I_i \subseteq I_j$. Thus R is a uniserial ring, hence local. As R has but finitely many ideals, R is Artinian and Noetherian. It follows that R is a special principal ideal ring, and so every ideal of R is a power of the nilpotent maximal ideal $\mathfrak{m} := \{x \in R\colon p^{k-1}x = 0\}$. Now suppose that $x, y \in R$ have the same additive order, which we may assume is greater than 1, and let i be greatest such that $x \in \mathfrak{m}^i$. As the I_j form the set of ideals of R and since x and y have the same additive order, it follows that i is greatest such that $y \in \mathfrak{m}^i$. Because R is uniserial, we may suppose without loss of generality that $xa = y$ for some $a \in R$. It now suffices to prove that a is a unit. If not, then $a \in \mathfrak{m}$. But then $y \in \mathfrak{m}^{i+1}$, contradicting the maximality of i.

Now for the general case. Let us suppose that $R \cong R_1 \times \cdots \times R_n$, where the R_i satisfy (1) and (2) above. Let us show that $R_1 \times \cdots \times R_n$ is order-associate. Let $\mathbf{x} := (x_1, \ldots, x_n)$ and $\mathbf{y} := (y_1, \ldots, y_n)$ be members of $R_1 \times \cdots \times R_n$ of the same additive order. then $|\mathbf{x}| = |x_1| \cdots |x_n| = |\mathbf{y}| = |y_1| \cdots |y_n|$. By the Fundamental Theorem of Arithmetic, $|x_i| = |y_i|$ for $1 \leq i \leq n$. By what we proved above, for each i, there is a unit $u_i \in R_i$ such that $x_i u_i = y_i$. Then $\mathbf{u} := (u_1, \ldots, u_n)$ is a unit of $R_1 \times \cdots \times R_n$ and $\mathbf{xu} = \mathbf{y}$, concluding the proof of sufficiency.

Now for the proof of necessity. Let R be a nontrivial order-associate ring. We consider two cases.

Case 1: R has characteristic 0. Then 1 has infinite additive order. But then for any positive integer n, $n := n \cdot 1$ has infinite additive order. Because R is order-associate, there is a unit $u \in R$ such that $1 \cdot u = n$, that is, n is a unit. We claim that $(R, +)$ is torsion-free. Indeed, suppose that $r \in R$ and that $nr = 0$ for some positive integer n. Because n is a unit, $r = 0$, proving the claim. Now let $r \in R$ be nonzero. Then 1 and r have the same additive order, so as above, r is a unit. This proves that R is a field, as desired.

Case 2: R has positive characteristic. First, we claim that $(R, +)$ is weakly T_1'. Toward this end, let $a, b \in R$ have the same additive order. Then there is a unit $u \in R$ such that $au = b$. Define $\varphi\colon R \to R$ by $\varphi(x) := xu$. It is easy to see that φ is an automorphism of $(R, +)$. Moreover, $b = au = \varphi(a)$. It now follows from Theorem 1 and the fact that R has positive characteristic that $(R, +) \cong \bigoplus_{p \text{ prime}} G_p$, where each G_p is a homococyclic p-group. Now, almost all G_p are trivial, since there is a finite bound on the additive orders of the members of R. So we have $(R, +) \cong G_{p_1} \oplus \cdots \oplus G_{p_n}$, where the p_i are distinct primes. Moreover, because of the finite bound on the additive orders of the members of R, $G_p \ncong \mathbb{Z}/p^\infty\mathbb{Z}$ for any prime p. It is easy to see that each G_{p_i} is closed under multiplication and that if $x \in G_{p_i}$ and $y \in G_{p_j}$ for $i \neq j$, then $xy = 0$. Setting $R_i := G_{p_i}$, $1 \leq i \leq n$, we see that $R \cong R_1 \times \cdots \times R_n$, and we have shown that (1) holds. It remains to prove (2). Observe that since R is order-associate, so is each R_i. Thus to complete the proof, it suffices without loss of generality to suppose that $n = 1$. So we suppose that $(R, +)$ is homocyclic of exponent p^k, $k > 0$. We must show that the ideals of R are given by $I_j := \{x \in R\colon p^j x = 0\}$ for $0 \leq j \leq k$. It is clear that each I_j is an ideal of R and that, moreover, the I_j are distinct. Let I be an arbitrary ideal of R. We will show that $I = I_j$ for some j, $0 \leq j \leq k$. Toward this end, let j be largest such that there is an element r^* of I of order p^j. We claim that $I = I_j$. It is immediate by maximality of j that $I \subseteq I_j$. Conversely, let $x \in I_j$. Then $|x| = p^i$ for some i, $0 \leq i \leq j$. Further, $p^{j-i}r^*$ has order

p^i. Because R is order-associate, there is a unit u such that $up^{j-i}r^* = x$. But then $x \in I$, as desired. □

Relative to the requisite additive structure imposed by Theorem 1, the order-associate rings are precisely the rings with the fewest possible number of ideals. Hence we may restate Theorem 2 as follows:

Corollary 4 (Restatement of Theorem 2)**.** *Let R be a ring. Then R is order-associate if and only if $(R, +)$ is torsion-free divisible or a finite direct sum of homocyclic p-groups, with the additional property that R has the fewest possible number of ideals of any ring whose additive group belongs to the class of groups defined above.*

The following realizability question follows naturally from our work above.

Question 4. Fix a prime number p and a positive integer k. Suppose that G is a homocyclic p-group of exponent p^k. Does there exist an order-associate ring R such that $(R, +) \cong G$?

The answer is affirmative, as show below.

Proposition 1. *Let G, p, and k be as stated above. There is an order-associate ring R such that $(R, +) \cong G$.*

Proof. We first consider a finite direct sum G of copies of $\mathbb{Z}/p^k\mathbb{Z}$; say there are $r > 0$ such summands. Let R be the Galois extension of $\mathbb{Z}/p^k\mathbb{Z}$ of degree r. Then (see [8], pp. 307–308) $R \cong \mathbb{Z}/p^k\mathbb{Z}[X]/\langle f\rangle$, where f is a basic monic irreducible polynomial in $\mathbb{Z}/p^k[X]$ of degree r (such polynomials exist for every $r > 0$). Because f is monic, we may apply the Division Algorithm to express every member of $\mathbb{Z}/p^k\mathbb{Z}[X]$ modulo $\langle f\rangle$ uniquely in the form $a_0 + a_1X + \cdots + a_{r-1}X^{r-1}$, where the a_is are integers mod p^k. It is immediate from this fact that $(R, +) \cong G$. It remains to show that R is order-associate. Toward this end, it is well-known (see [8], p. 308) that the ideals of R are precisely $\langle p^i\rangle$, where $0 \leq i \leq k$. Hence R has precisely $k + 1$ ideals. As the $k + 1$ ideals stated in (2) of Proposition 2 are distinct, it follows that they form precisely the the collection of ideals of R. Hence by Theorem 2, R is order-associate.

Now suppose that G is an infinite direct sum of copies of $\mathbb{Z}/p^k\mathbb{Z}$. We will invoke some basic model theory to handle this case; for background, we refer the reader to the classic text [1]. Let $\mathfrak{L}$ be the language of rings. Consider the axioms for a commutative ring with identity, along with the following assertions, all expressible in first-order logic in $\mathfrak{L}$:

(1) for every x, $p^kx = 0$,
(2) there exists an element of additive order p^k, and
(3) for $0 \leq i \leq k$, a sentence asserting that for all x, y: if x and y have order p^i, then there is a unit u such that $xu = y$.

From our work above, there are arbitrarily large finite models of the above sentences. By the Compactness Theorem, there is an infinite model. By Lowenheim-Skolem, there are models of every infinite cardinality. Invoking Theorem 2, it follows that $G \cong (R, +)$ for some order-associate ring R, and the proof is complete. □

4.2. **Weakly T_1' rings.** *Throughout the rest of this note, by "ring", we mean only an associative ring, but we do not assume an identity or commutativity unless stated as such.* Our final query is the following.

Question 5. Can the rings R be classified with the property that for any elements $a, b \in R$ of the same additive order, there is a ring automorphism φ of R for which $\varphi(a) = b$?

As with groups, let us agree to call such rings **weakly $\mathbf{T_1'}$**. While we do not completely solve the question, we obtain some partial results. Toward this end, we begin with several lemmas.

Lemma 4. *Let R be a weakly T_1' ring. Then $(R, +) \cong D \oplus \bigoplus_{p \text{ prime}} G_p$, where D is a torsion-free divisible abelian group, and each G_p is a homococyclic p-group.*

Proof. Immediate from Theorem 1. □

Lemma 5. *Let R be a ring with identity $1 \neq 0$. Then R is weakly T_1' if and only if $R \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. It is clear that $\mathbb{Z}/2\mathbb{Z}$ is weakly T_1'. Conversely, suppose that R is a unital ring with $1 \neq 0$ which is weakly T_1'. Note that for any ring automorphism φ of R, we must have $\varphi(1) = 1$. Now, if R has characteristic 0, then 1 has infinite additive order, and thus so does $1 + 1 = 2$. Hence there is a ring automorphism φ of R such that $\varphi(1) = 2$, a contradiction. Hence R has finite characteristic $n > 1$. Suppose that $n \geq 3$. Then both 1 and $n - 1$ have additive order n, and so again, there is a ring automorphism φ of R for which $\varphi(1) = n - 1$, and we have another contradiction. Hence R has characteristic 2. Let $x \neq 0$. Then x and 1 have the same additive order, and so there is a ring automorphism φ of R for which $1 = \varphi(1) = x$. This proves that $R \cong \mathbb{Z}/2\mathbb{Z}$, as desired. □

For the purposes of the next lemma, let us call a ring R **strongly** T_1' provided that for any nonzero elements $a, b \in R$, there is a ring automorphism of R mapping a to b. Before continuing, recall that if G is an abelian group, then G becomes a ring by defining $x \cdot y = 0$ for every $x, y \in G$; we denote this ring by G_0.

Lemma 6. *Let R be a finite ring with more than one element. Then R is strongly T_1' if and only if R is a finite direct product of copies of $(\mathbb{Z}/p\mathbb{Z})_0$, p a prime, or $R \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. It is clear that $\mathbb{Z}/2\mathbb{Z}$ is strongly T_1'. Now suppose that R is a finite direct product of copies of $(\mathbb{Z}/p\mathbb{Z})_0$, p a prime. Then $(R, +)$ is naturally a vector space over $\mathbb{Z}/p\mathbb{Z}$. Now let $a, b \in R$ be nonzero. Then both $\{a\}$ and $\{b\}$ are linearly independent, hence can be extended to bases β and β', respectively, for $(R, +)$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Let $\varphi\colon \beta \to \beta'$ be a bijection for which $\varphi(a) = b$. Then φ extended uniquely to a vector space automorphism $\overline{\varphi}$ of $(R, +)$. But then $\overline{\varphi}$ is also an abelian group automorphism. Since the multiplication on R is trivial, $\overline{\varphi}$ is a ring automorphism.

Conversely, suppose that R is a finite ring with more than one element which is strongly T_1'. It is immediate that any two nonzero elements of R have the same additive order. Since R is finite, R has an element of additive order p for some prime p. Hence $(R, +)$ is isomorphic to a direct sum of copies of $\mathbb{Z}/p\mathbb{Z}$. If the multiplication on R is trivial, we are done. So assume the multiplication is nontrivial. Next, we make several claims.

$$R^2 = R. \tag{4.1}$$

It suffices to prove that $R \subseteq R^2$. Since the multiplication on R is nontrivial, there are elements $x, y \in R$ for which $xy \neq 0$; let $xy = z$. Now let $a \in R$ be an arbitrary nonzero element. There

is a ring automorphism φ of R for which $\varphi(z) = a$. But then $\varphi(x)\varphi(y) = a$, and so $a \in R^2$. This proves (4.1).

(4.2) R has no nonzero nilpotent elements.

Suppose by way of contradiction that there is a nonzero $r \in R$ which is nilpotent. Then every element of R is nilpotent. Now, let M be a maximal ideal of R (which exists as R is finite and nontrivial). We claim that M is prime (this is well-known, but we give the short argument). Indeed, suppose by way of contradiction that there exist $x, y \in R$ such that $xy \in M$ but $x \notin M$ and $y \notin M$. Then $(M, x) = (M, y) = R$, where (M, x) and (M, y) are the ideals of R generated by M and x and M and y, respectively. But then $(M, x)(M, y) = R^2 = R$ by (4.1) above. However, $(M, x)(M, y) \subseteq M$ since $xy \in M$. This is a contradiction. Hence M is a prime ideal. But since every member of R is nilpotent, it follows that $M = R$, a contradiction, which proves (4.2).

We are now equipped to conclude the proof. Let $x \in R$ be nonzero. Since R is finite, $x, x^2, x^3, \ldots$ cannot all be distinct. Hence there are integers i and j with $1 \leq i < j$ such that $x^i = x^j$. By induction, it follows that $x^i = x^{n(j-i)+i}$ for every positive integer n. So without loss of generality, we may assume that $i < \frac{j}{2}$. Now let $k := j - 2i > 0$. Observe that $(x^{i+k})^2 = x^{2i+2k} = x^{j+k} = x^j x^k = x^i x^k = x^{i+k}$. Set $y := x^{i+k}$. By (4.2), y is nonzero. Moreover, $y^2 = y$. Because R is strongly T_1', every element of R is idempotent, that is, R is Boolean. It is well-known that the only finite Boolean rings are finite direct products of copies of $\mathbb{Z}/2\mathbb{Z}$. In particular, R has an identity. Hence by Lemma 5, $R \cong \mathbb{Z}/2\mathbb{Z}$, as desired. □

We are now equipped to state a necessary condition in order for a finite ring to be weakly T_1'. We leave open whether all finite weakly T_1' rings can be classified (and, more generally, all weakly T_1' rings). Recall first that a ring R is said to be a **nilring** provided every $r \in R$ is nilpotent.

Proposition 2. *Let R be a nonzero finite ring which is weakly T_1'. Then either R is a nilring or $R \cong N \times \mathbb{Z}/2\mathbb{Z}$ for some weakly T_1' nilring N.*

Proof. Let $(R, +, \odot)$ be a nonzero finite weakly T_1' ring. As is well-known (see the proof of Theorem 2), there are distinct primes $p_1, \ldots, p_n$ for which $R = R_1 \times \cdots \times R_n$, where each R_i is a nonzero ring of order a power of p_i. It is easy to see that each R_i is also a weakly T_1' ring. Fix i with $1 \leq i \leq n$, and set $S := R_i$ and $p := p_i$. Then by Lemma 4, there exist positive integers r and k such that $(S, +)$ has rank r and exponent p^k, that is,

$$(S, +) \cong \bigoplus_r \mathbb{Z}/p^k\mathbb{Z}. \tag{4.3}$$

We now analyze the structure of the ring S by considering two cases.

Case 1: $r = 1$ and $p = 2$. Then $(S, +) \cong \mathbb{Z}/2^k\mathbb{Z}$. Next, set $\overline{1} \odot \overline{1} := \overline{\alpha}$ (note that the bar indicates reducing modulo 2^k). Then for any integers a and b, it follows that $\overline{a} \odot \overline{b} = \overline{\alpha ab}$ (and hence S is a commutative ring). If α is even, then it is easy to see that every $\overline{a} \in S$ is nilpotent, and hence S is a nilring. Now suppose that α is odd. Then there is an integer β such that

$\overline{\alpha} \cdot \overline{\beta} = \overline{1}$. Now let $\overline{a} \in S$ be arbitrary. Then observe that $\overline{\beta} \odot \overline{a} = \overline{\alpha}\overline{\beta}\overline{a} = \overline{a}$. Thus $\overline{\beta}$ is an identity for S. Lemma 5 implies that $S \cong \mathbb{Z}/2\mathbb{Z}$.

Case 2: $r > 1$ or $p \neq 2$. We now proceed by induction on k (relative to (4.3)) that S is a nilring for every positive integer k. In the base case, we take S to be a weakly T_1' ring such that $(S, +) \cong \bigoplus_r \mathbb{Z}/p\mathbb{Z}$. Because $r > 1$ or $p \neq 2$, Lemma 6 implies that $xy = 0$ for all $x, y \in S$. Hence S is a nilring in this case. For the inductive step, let k be a positive integer and assume that for any weakly T_1' ring T such that $(T, +) \cong \bigoplus_r \mathbb{Z}/p^k\mathbb{Z}$, T is a nilring. Now suppose that A is a weakly T_1' ring and that $(A, +) \cong \bigoplus_r \mathbb{Z}/p^{k+1}\mathbb{Z}$. Next, set $I_k := \{a \in A \colon p^k a = 0\}$. It is clear that I_k is an (two-sided) ideal of A (thus a subring of A) and that $(I_k, +) \cong \bigoplus_r \mathbb{Z}/p^k\mathbb{Z}$. Suppose now that $x, y \in I_k$ have the same additive order. Then there is an automorphism φ of A for which $\varphi(x) = y$. Now, it is clear that $\varphi(I_k) \subseteq I_k$. As I_k is finite and φ is one-to-one, we see that $\varphi(I_k) = I_k$. We deduce that I_k is a weakly T_1' ring. By the inductive hypothesis, I_k is a nilring. Next, let B be the ring defined by $B := A/I_k$. It is patent that $(B, +) \cong \bigoplus_r \mathbb{Z}/p\mathbb{Z}$. We claim that B is a weakly T_1' ring. Consider two elements $\overline{x}, \overline{y} \in B$ of the same additive order (here, barring means taking x and y modulo the ideal I_k). If the order is 1, then $\overline{x} = \overline{y} = \overline{0}$, and the identity map is a ring automorphism of B mapping $\overline{x}$ to $\overline{y}$. Now suppose that $\overline{x}$ and $\overline{y}$ have order p. Then $p^k x \neq 0$, $p^k y \neq 0$, but $p^{k+1}x = p^{k+1}y = 0$. We deduce that x and y have additive order p^{k+1}. Because A is weakly T_1', there is a ring automorphism $\varphi \colon A \to A$ such that $\varphi(x) = y$. Now define $\overline{\varphi} \colon B \to B$ by $\overline{\varphi}(\overline{a}) = \overline{\varphi(a)}$. Observe that if $c - d \in I_k$, also $\varphi(c-d) = \varphi(c) - \varphi(d) \in I_k$. This shows that $\overline{\varphi}$ is well-defined. It is clear that $\overline{\varphi}$ is onto, as φ is onto. As B is finite, $\overline{\varphi}$ is also injective. It is now clear that $\overline{\varphi}$ is a ring automorphism of B, and thus B is a weakly T_1' ring. It is now immmediate from Lemma 6 that the multiplication on B is trivial. Hence for any $a \in A$, we have $a^2 \in I_k$. By the inductive hypothesis, a^2 is nilpotent, and thus so is a. This proves that A is a nilring, and completes the induction.

Finally, recall from the beginning of the proof that we have $R = R_1 \times \cdots \times R_n$, where each R_i is a nonzero weakly T_1' ring of order a power of the prime p_i. If every R_i is a nilring, then R is a nilring and we are done. Now suppose without loss of generality that $R_1 \cong \mathbb{Z}/2\mathbb{Z}$. If $n = 1$, we can choose N be the trivial ring to get $R_1 \cong N \times \mathbb{Z}/2\mathbb{Z}$, with N trivially weakly T_1'. If $n > 1$, then we see that $R \cong \mathbb{Z}/2\mathbb{Z} \times T$, where (from our work above in Case 2) T is a weakly T_1' nilring. The proof is now complete. □

As an illustration of the previous theorem, we classify all weakly T_1' rings with (necessarily finite, by Lemma 4) cyclic additive group.

Proposition 3. *Let R be a finite ring with cyclic additive group. Then R is weakly T_1' if and only if the following hold:*

(1) *$R \cong S \times T$ for some rings S and $(T, +, \odot)$ such that*
(2) *$S \cong (\mathbb{Z}/m\mathbb{Z})_0$ for some positive odd integer m, and*
(3) *$(T, +) = \mathbb{Z}/2^k\mathbb{Z}$ for some non-negative integer k. Moreover, if $k \geq 2$, then $\overline{1} \odot \overline{1} = \overline{0}$ or $\overline{1} \odot \overline{1} = 2^{k-1}\overline{\alpha}$ for some odd integer α.*

Proof. Consider first a finite ring R with cyclic additive group which satisfies (1)–(3) above. Note that $|S|$ and $|T|$ are relatively prime, and so it suffices to prove that both S and T are weakly T_1' rings. It is immediate from Theorem 1 and the Chinese Remainder Theorem that S is a weakly T_1' ring. Now consider T. If $k = 0$ or $k = 1$, it is clear that T is a weakly T_1' ring.

Now suppose that $k \geq 2$. If $\overline{1} \odot \overline{1} = \overline{0}$, then it is immediate that $\overline{a} \odot \overline{b} = \overline{0}$ for all integers a and b, and hence T is a weakly T_1' ring in this case, as above. Now suppose that $\overline{1} \odot \overline{1} = 2^{k-1}\overline{\alpha}$ for some odd integer α. Let $\overline{x}, \overline{y} \in T$ have the same additive order. Because $\mathbb{Z}/2^k\mathbb{Z}$ is a weakly T_1' abelian group, there is a group automorphism $\varphi\colon T \to T$ such that $\varphi(\overline{x}) = \overline{y}$. It remains to show that φ preserves multiplication. For any integers a and b, we have $\varphi(\overline{a} \odot \overline{b}) = \varphi(\overline{1})(\overline{a} \odot \overline{b}) = ab2^{k-1}\alpha\varphi(\overline{1})$. Moreover, $\varphi(\overline{a}) \odot \varphi(\overline{b}) = \varphi(\overline{1})\overline{a} \odot \varphi(\overline{1})\overline{b} = ab2^{k-1}\alpha\varphi(\overline{1})^2$ (the square is now taken relative to the usual multiplication in $\mathbb{Z}/2^k\mathbb{Z}$). Hence it suffices to prove that $ab2^{k-1}\alpha\varphi(\overline{1}) = ab2^{k-1}\alpha\varphi(\overline{1})^2$, that is, that

$$ab2^{k-1}\alpha(\varphi(\overline{1}) - \varphi(\overline{1})^2) = \overline{0}. \tag{4.4}$$

Note that $\varphi(\overline{1}) - \varphi(\overline{1})^2 = \overline{m}$ for some even integer m. As 2^k kills T, (4.4) follows.

Conversely, suppose that R is a finite weakly T_1' ring with cyclic additive group. Without loss of generality, we may assume that $|R| := m > 1$; let $m = p_1^{a_1} \cdots p_k^{a_k}$ be the prime factorization of m. As we have seen (and as is well-known), $R \cong R_1 \times \cdots \times R_k$ for some weakly T_1' rings R_i of cardinality $p_i^{a_i}$. Fix i with $1 \leq i \leq k$, and set $U := (R_i, +, \odot)$, $p := p_i$, and $a := a_i$. We now analyze the structure of U, then conclude the proof. First, note that since $(R, +)$ is cyclic, we may assume that $(U, +) = \mathbb{Z}/p^a\mathbb{Z}$. Let $\overline{1} \odot \overline{1} = p^j\overline{\alpha}$, where $0 \leq j \leq a$ and $(\alpha, p) = 1$ (where we reduce mod p^a as previously).

Case 1: p is odd. As $\overline{1}$ and $-\overline{1}$ have the same additive order in U, there is a ring automorphism $\varphi\colon U \to U$ for which $\varphi(\overline{1}) = -\overline{1}$. Now, $\varphi(\overline{1} \odot \overline{1}) = \varphi(\overline{1}) \odot \varphi(\overline{1})$. It follows as in (4.4) above that (setting $\overline{a} = \overline{b} = \overline{1}$) $p^j\alpha(\varphi(\overline{1}) - \varphi(\overline{1})^2) = \overline{0}$. Since $\varphi(\overline{1}) = -\overline{1}$, the previous equation reduces to $2p^j\overline{\alpha} = \overline{0}$. Because p is odd, this equation becomes $p^j\overline{\alpha} = \overline{0}$. Because $(\alpha, p) = 1$, we deduce that $j = a$. Thus $\overline{1} \odot \overline{1} = \overline{0}$, from which it follows that $\odot$ is trivial on U.

Case 2: $p = 2$. With the same argument given in Case 1 above, we see that $2^{j+1}\overline{\alpha} = \overline{0}$. Because α is odd, it follows that either $j = a$ or $j + 1 = a$. From this, we see that (3) is satisfied for the ring U.

As $1 \leq i \leq k$ was arbitrary, our work above concludes the proof. □

We conclude the article with several more examples. The first example shows that we cannot strengthen Proposition 2 to an if and only if statement.

Example 1. *For any finite homocyclic group G of rank greater than one, there is a nilring which is not weakly T_1' but with additive group isomorphic to G.*

Proof. Let p be a prime and let k and r be positive integers, with $r > 1$. Now let $G := \bigoplus_r \mathbb{Z}/p^k\mathbb{Z}$ be the direct sum of r copies of the group $\mathbb{Z}/p^k\mathbb{Z}$. Next, set $R := X\mathbb{Z}/p^k\mathbb{Z}[X]/\langle X^{r+1}\rangle$, where $X\mathbb{Z}/p^k\mathbb{Z}[X]$ is the ring of polynomials in the variable X over the ring $\mathbb{Z}/p^k\mathbb{Z}$ with constant term zero. It is clear that $(R, +) \cong G$ and that, moreover, R is a nilring. Finally, note that $\overline{X}$ and $\overline{X}^r$ have additive order p^k but $\overline{X}$ has nilpotency at least three whereas $\overline{X}^r$ has nilpotency two. So there can be no ring automorphism sending one of these elements to the other, proving that R is not a weakly T_1' ring. □

Though $\mathbb{Z}/2\mathbb{Z}$ is the only finite Boolean weakly T_1' ring, Boolean weakly T_1' rings exist of every infinite cardinality.

Example 2. *Let κ be an infinite cardinal. Then there is a Boolean weakly T_1' ring of cardinality κ.*

Proof. Let $\{X_i\colon i < k\}$ be a collection of κ-many indeterminates. Next, consider the ring $S := \mathbb{F}_2[X_i\colon i < \kappa]/\langle X_i^2 - X_i\colon i < \kappa\rangle$. It is routine to check that this ring is Boolean. Let R be the subring of S consisting of all members of S with constant term 0. Then R is a non-unital Boolean ring. Further, any map of $\{X_i\colon i < \kappa\}$ (modulo the ideal $\langle X_i^2 - X_i\colon i < \kappa\rangle$) into a Boolean ring T may be extended to a homomorphism of R into T. Thus R is a free non-unital Boolean ring. By Propositions 9.13 and 9.14 of [6], R is a weakly T_1' Boolean ring. □

Remark 2. One can also construct infinite Boolean weakly T_1' rings via a different construction. To wit, let κ be an uncountable cardinal and let S be a set of cardinality κ. Now let $\mathcal{C}$ denote the collection of countable subsets of S. Define a relation $\sim$ on $\mathcal{C}$ by $A \sim B$ if and only if $(A\backslash B) \cup (B\backslash A)$ is finite. Then $\sim$ is an equivalence relation on $\mathcal{C}$. For a countable subset A of S, let $[A]$ denote its equivalence class. Then $R := \mathcal{C}/\sim$ becomes a Boolean ring via the well-defined operations $[A] + [B] := [(A\backslash B) \cup (B\backslash A)]$ and $[A] \cdot [B] := [A \cap B]$. Now suppose that $[A], [B]$ are nonzero (that is, $[A] \neq [\varnothing]$ and $[B] \neq [\varnothing]$). Then A and B are countably infinite. Let $f\colon A \to B$ be a bijection, and extend f to a permutation of S (this is possible as S is uncountable). Now define $\overline{f}\colon R \to R$ by $\overline{f}([X]) := [f(X)]$. Then $\overline{f}$ is well-defined, and it is immediate that $\overline{f}([A]) = [B]$. One checks easily that $\overline{f}$ is an automorphism of R, as desired.

Unlike the previous example, this construction does not produce examples of Boolean weakly T_1' rings of every uncountable cardinality in ZFC. For κ as above, one can show that $|R| = \kappa^{\aleph_0}$. Under the Generalized Continuum Hypothesis, $\aleph_0^{\aleph_0} = \aleph_1$, $\aleph_n^{\aleph_0} = \aleph_n$ for $0 < n < \omega$, and $\aleph_\omega^{\aleph_0} > \aleph_\omega$ due to König's Theorem. Hence, in ZFC, the above construction does not produce a Boolean weakly T_1' ring of cardinality $\aleph_\omega^{\aleph_0}$. We refer the reader to [5] for background in axiomatic set theory.

5. An open problem

We close the paper by stating the following problem for further research.

Problem 1. Classify the weakly T_1' rings.

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