MATH 2150 Autumn 2022 Lecture 8

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(2) If it is NOT the case that $(\alpha \vee \beta)$ is true, then it is not the case that AT LEAST ONE of α and β is true. Therefore, BOTH are false (similarly, the right side implies the left side).

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Negate the formula $(p \land (\neg q))$ so that every negation symbol immediately precedes a variable.

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(a) $(\neg \forall x P(x)) \equiv \exists x(\neg P(x))$, *and*

(b) $(\neg \exists x P(x)) \equiv \forall x(\neg P(x))$.

I have not rigorously defined the symbol "$\equiv$" in predicate logic (I've only done so in propositional logic).

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