Math 2150 Notes - June 21, 2021

The lecture material will loosely follow section 1.3 of the Rosen text. As you know, we have been studying propositional logic since the beginning of the semester. This is a very natural and basic logic (and very important in computer science and even engineering), but it is simply too primitive to capture the essence of most mathematical statements. For this, we need a more robust (and this means, unfortunately, more complicated) logic, often called **predicate logic** or **first order logic**. I like Rosen's convention to introduce this logic slowly and I will follow his convention.

Definition 1. *A **one-place predicate** is an assertion involving a single variable. If the variable is (say) x, then we often denote the predicate by "$P(x)$".*

IMPORTANT: variables will no longer be propositional; more on this shortly.

Example 1. *Let $P(x)$ be "$x = 2$".*

Note that "$x = 2$" is *not a proposition* (its truth value is dependent upon what x is; if no x is specified, then "$x = 2$" could be true (if $x = 2$) or false (if $x \neq 2$).

Example 2. *Let $Q(y)$ be "Person y is over 6 feet tall."*

Again, observe that $Q(y)$ is not a proposition, but *it becomes a proposition once a particular object is substituted for y.*

Remark 1. *Throughout our study of predicate logic, capital letters will in general denote predicates.*

Example 3. *Let $P(z)$ be "$z > 2$" (z is greater than 2). Find $P(0)$, $P(2)$, and $P(3)$ along with their truth values.*

Solution Before we begin, let's briefly digress back to basic algebra. Consider the function $f(x) = x^2$. How do you find $f(5)$? Well, you just "plug in" 5 for x to get $f(5) = 5^2 = 25$. We do the same thing here, but instead of getting a real number as an output, we get a *proposition* as an output. Since $P(z)$ is "$z > 2$", to find $P(0)$, we plug in 0 for z to obtain $P(0) =$ "$0 > 2$", which is false. Similarly, $P(2)$ is "$2 > 2$" which is also false. Finally, $P(3)$ is "$3 > 2$", which is true. $\square$

Analogous to the domain of a function, we can also specify so-called domains for one-place predicates.

Definition 2. *Let $P(x)$ be a one-place predicate. A **domain** for the variable x is a specified collection of values which may be substituted for x.*

Example 4. *Let $P(x)$ be "x is over 3 years old." Then it may be reasonable to let the domain for x be the collection of all people, or the collection of all dogs, or the collection of all children, etc.*

Remark 2. *Suppose that $P(x)$ is a predicate. In general, there will be infinitely many domains for x, so please do NOT assume that there can only be one; the domain often depends upon the context. More on this later.*

We began our study of propositional logic by introducing the logical operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. *Propositional logic will continue to play a role in predicate logic, so you can't forget what you learned.* We now augment the operators of propositional logic by adding so-called *quantifiers*, defined below.

Definition 3. *In predicate logic, we augment the language of propositional logic by introducing the **universal** and **existential quantifiers**, defined below. For both, let $P(x)$ be a one-place predicate.*

1. *The symbol $\forall$ is called the universal quantifier, and it translates to "for every" or "for all", and*
2. *the symbol $\exists$ is called the existential quantifier, and it translates to "there exists".*

The next examples will be very important to get down.

Example 5. *Let $P(x)$ be "$x > 0$". Then the expression $\forall x P(x)$ translates to "For all x, x is greater than 0." If a domain is specified, the translation is "For all x in the specified domain, x is greater than 0". So suppose we let the domain of x be the collection of all real numbers. Relative to this domain, the translation is "For all real numbers x, x is greater than 0." If we change the domain to the set of all whole numbers, then the translation is "For all whole numbers x, x is greater than 0."*

Example 6. *Let $P(x)$ be "x is even". Then $\exists x P(x)$ translates to "There exists x such that x is even." If a domain is specified, the translation is "There exists x in the specified domain such that x is even". So suppose we let the domain of x be the collection of all integers. Relative to this domain, the translation is "There exists an integer x such that x is even". If we change the domain to the set of positive integers, then the translation is "There exists a positive integer x such that x is even".*

One of the main take-aways here is the following: translate $\forall x$ as "for all (whatever type of object) x" and $\exists x$ as "there exists (whatever type of object) x such that".

We will return to semantics and translations shortly. But first, I want to introduce another algorithm for churning out formulas in this new logic. You should see that the algorithm is similar to the one used in propositional logic, but a bit more complicated.

Definition 4 (Recursive definition for formulas involving one-place predicates)**.** *The following rules govern the construction of formulas in predicate logic with one-place predicates:*

1. *Every one-place predicate $P(x)$ is a formula (be aware that you'll see various upper and lower case letters here; the upper case letters will denote one-place predicates and the lower case letters will denote variables).*
2. *If α and β are formulas, so are $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$.*
3. *If α is a formula, then so are $\forall x\alpha$ and $\exists x\alpha$ (your variable could be some other lower case letter, by the way).*
4. *No string is a formula unless it is compelled to be so by repeated application of (1)–(3) above.*

For now, let's focus exclusively on syntax. As with propositional logic, I do NOT want you to adopt conventions for reducing the number of parentheses.

Example 7. *Justify that $\forall x(P(x) \to Q(y))$ is a formula.*

Solution $P(x)$ and $Q(y)$ are formulas by (1) since they are one-place predicates. Now by (2), $(P(x) \to Q(y))$ is a formula. By (3), $\forall x(P(x) \to Q(y))$ is a formula. □

Example 8. *Justify that $\exists x \forall y(P(x) \vee (\neg R(y)))$ is a formula.*

Solution $R(y)$ is a formula by (1) since it is a one-place predicate. By (2), $(\neg R(y))$ is also a formula. By (1) again, $P(x)$ is a formula since it is a one-place predicate. By (2), $(P(x) \vee (\neg R(y)))$ is a formula (since we have justified that both $P(x)$ and $(\neg R(y))$ are formulas). By (3), $\forall y(P(x) \vee (\neg R(y)))$ is a formula. Now by (3) again, $\exists x \forall y(P(x) \vee (\neg R(y)))$ is a formula. □

Next, let's venture back to semantics for a bit to conclude the lecture for today. In any formula you give as a homework answer, you should strive to make it syntactically sound by following the above recursive algorithm (again, please do NOT omit parentheses).

Example 9. *Let $P(x)$ be "x is even" and let the domain D for x be the set $\{1, 2, 3, 4, 5, \ldots\}$ of all positive integers.*

1. *Is $\forall x P(x)$ true relative to D?*
2. *Is $\exists x P(x)$ true relative to D?*

Solution Consider the above $P(x)$ and D.

1. First, I suggest translating the statement into English as we did in earlier examples. The translation is: "For every positive integer x, x is even". In other words, the assertion is that *no matter what* positive integer x represents, x is even. This is false: 1 is a positive integer that is NOT even.

2. Again, let's translate first: "There exists a positive integer x such that x is even". In other words, the assertion is that *there is at least one positive integer* you can substitute in for x making x even. And this is true: if we substitute (say) 6 in for x, then x is even. $\square$

Example 10. *Let $P(x)$ be "x is a professional basketball player". Find a domain for x making $\forall x P(x)$ true and another domain for x making $\forall x P(x)$ false.*

Solution Let $P(x)$ be as above.

1. To make $\forall x P(x)$ true, we may take D to simply be the collection of all professional basketball players. Relative to this domain, the formula $\forall x P(x)$ translates to "For every professional basketball player x, x is a professional basketball player", which is obviously true.

2. To make $\forall x P(x)$ false, we may take D to be the set of all students in my 2150 section. Certainly not all of you are professional basketball players (in fact, I'm sure none of you are, and so $\exists x P(x)$ is also false relative to this domain). $\square$

We now come upon another recursive definition; here is the motivation: consider the predicate $P(x)$ "$x > 0$" relative to the domain D of all integers $(\cdots, -3, -2, -1, 0, 1, 2, 3, \ldots)$. Now let's look at the formula $\forall x P(x)$ relative to this domain. Observe that this formula translates to "For every integer x, $x > 0$." Now, observe that *this is a false proposition*; **we did not need to plug anything in for x to see this.** But what if we omitted the "$\forall x$" from the formula? Then we would be left with just "$x > 0$". Observe that relative to the domain D of integers, this is **not a proposition**, since the assertion is true for some integers and false for others. Very roughly, what is going on is the following: In the formula $\forall x P(x)$, then "x" was *quantified* whereas the x is not quantified in $P(x)$. IMPORTANT!!!: Throughout our study of predicate logic, whenever you translate an English sentence into a formulas, ALL variables will ALWAYS be quantified in your formula. I can't overstate the importance of remembering this. To assist, I am going to give you another recursive definition.

Definition 5 (Recursive definition of "freeness")**.** *Let x and y be (possibly the same) variables and α and β be formulas.*

1. *x occurs free in any 1-place predicate $P(x)$,*
2. *x occurs free in $(\neg\alpha)$ exactly when x occurs free in α,*
3. *x occurs free in $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$, $(\alpha \to \beta)$, and $(\alpha \leftrightarrow \beta)$ exactly when x occurs free in α or β,*
4. *x occurs free in $\exists y \alpha$ exactly when x occurs free in α and $x \neq y$, and*
5. *x occurs free in $\forall y \alpha$ exactly when x occurs free in α and $x \neq y$.*[1]

Example 11. *Using the above definition, justify why x occurs free in $P(x)$.*

Solution This is easy: immediate from (1). $\square$

[1] Thus x NEVER occurs free in $\forall x \alpha$ or $\exists x \alpha$.

Example 12. *Using the above definition, justify why x occurs free in $(P(x) \vee Q(y))$.*

Solution x occurs free in $P(x)$ by (1) of the definition. Thus x occurs free in $(P(x) \vee Q(y))$ by (3) of the definition. $\square$

Example 13. *Using the above definition, justify why y occurs free in $(P(x) \rightarrow (\neg Q(y)))$.*

Solution y occurs free in $Q(y)$ by (1), and so y occurs free in $(\neg Q(y))$ by (2). Now by (3), y occurs free in $(P(x) \rightarrow (\neg Q(y)))$. $\square$

Example 14. *Using the above definition, justify why x occurs free in $(\forall x P(x) \rightarrow Q(x))$.*

Solution By (1), x occurs free in $Q(x)$. Thus by (3), x occurs free in $(\forall x P(x) \rightarrow Q(x))$. $\square$

Remark 3. *IMPORTANT!!!: Note that if we changed the formula in the previous example to $\forall x(P(x) \rightarrow Q(x))$, then x does NOT occur free (see footnote on previous page).*

Now, let's look at some example of translating English sentences into formulas. Remember what I said above: if you do this correctly, NO variable will EVER occur free!

Example 15. *Let $P(x)$ be "$x > 0$" and $Q(x)$ be "$x^2 > 0$". Translate "For every x, if $x > 0$, then $x^2 > 0$".*

Solution Note that I didn't give you a domain. The reason is that the domain isn't relevant for this problem. The formula is the following: $\forall x(P(x) \rightarrow Q(x))$. Observe that x does not occur free in this formula since there is quantification on the left which applies to the entire string following it. $\square$

Example 16. *Let $P(x)$ be "x is tall" and $Q(x)$ be "x plays basketball". Translate: "There is a person that is tall that doesn't play basketball".*

Solution $\exists x(P(x) \wedge (\neg Q(x)))$. Again, note that x does not occur free in this formula. Note also that I am using the correct syntax (recall the recursive definition of "formula" from the last set of notes). Finally, let's practice a skill I introduced last week: how would you translate $\exists x(P(x) \wedge (\neg Q(x)))$ to English? (Note that I've implicitly introduced the set of people as the domain for this problem) The translation is, "There exists a person x such that x is tall and x doesn't play basketball." If we de-geek this sentence and make it more natural, we get the English sentence above that we had to translate into a formula. $\square$

Example 17. *Let $P(x)$ be "x is even" and let $Q(x)$ be "$x + 2$ is even". Translate "If x is an even integer, then $x + 2$ is an even integer."*

Solution OK, here is where things get a bit more difficult. For concreteness, let us let the domain for x be the set of integers. Note that you don't see "for all", "for every", or "there is/there exists". It may be tempting to translate the above sentence as $(P(x) \rightarrow Q(x))$. But if you've processed what I said above, YOU SHOULD ALREADY KNOW THAT THIS CAN'T BE CORRECT SINCE THE VARIABLE x OCCURS FREE IN THIS FORMULA!!! What's missing is quantification at the front. Toward this end, the main idea is the following: there was no restriction placed upon the integer x; x can be ANY integer. So the way to think about this is to make the "invisible" universal quantifier at the beginning visible: let's rewrite the sentence as follows. "For every integer x, if x is even, then $x + 2$ is even." Now our translation becomes $\forall x(P(x) \rightarrow Q(x))$. $\square$

Example 18. *Let $R(x)$ be "x is a rose" and let $r(x)$ be "x is red".*

1. *Translate "There exists a red rose." if the domain for x is the collection of all roses.*
2. *Translate "Every rose is red." if the domain for x is the collection of all roses.*

Solution (1) If the domain is the collection of roses, then the predicate $R(x)$ above is irrelevant. So the translation becomes $\exists x r(x)$. Again, let's go backwards using the skill I taught in the previous set of notes: the translation is "There exists a rose x such that x is red", which then reduces to the sentence I gave you above, more colloquially.

(2) Again, $R(x)$ is irrelevant. So the translation is $\forall x r(x)$. Again, going backwards, the translation is "For every rose x, x is red." Writing this more naturally gives us the sentence in (2) above. □

Let us end by doing the previous problem, but with a broader domain (here, the predicate $R(x)$ will not be redundant).

Example 19. *Let $R(x)$ and $r(x)$ be as above, but now let's assume that the domain for x is the collection of all plants.*

1. *Translate "There exists a red rose." relative to this domain.*
2. *Translate "Every rose is red." relative to this domain.*

Solution (1) Note that if we just write $\exists x r(x)$, then the translation is "There exists a plant x such that x is red." But this is clearly NOT what we want to say; we want to say something more specific, namely, that there exists a *rose* that is red. The way to fix this is to specify that our x must also be a rose. So we get $\exists x(r(x) \wedge R(x))$. Again, going backward using the notes from last week, the translation is "There exists a plant x such that x is red and x is a rose." Again, this de-geeks to "There exists a red rose."

(2) This one is a bit tricky. For many of you, your gut will lead you astray here, so it is very important to try to digest what follows. Before giving you the correct answer, I'm going to give an answer which is not correct, but that some of you would come up with if I posed this in the classroom. Probably the most common answer I receive is this: $\forall x(R(x) \wedge r(x))$. But this isn't correct. Let's translate this using last week's notes: "For every plant x, x is a rose and x is red." But this is definitely MUCH stronger than what we are trying to express: what we are trying to do is say something about ALL ROSES, not ALL PLANTS. The way to fix this is to use the following observation: every rose is, in fact, a plant. So the way around this dilemma is to say that IF a plant x happens to be a rose, THEN x is red. So our formula is $\forall x(R(x) \to r(x))$. □