Math 2150 Assignment 4 Solutions

(1) Let $P(x, y, z)$ be the 3-place predicate "x has y dollars invested in z" and let $Q(a, b)$ be the 2-place predicate "$\frac{a}{b} = 1$".

(a)[2 pts] What is $P(Sara, 200, Amazon)$?

Solution. $P(Sara, 200, Amazon)$ is "Sara has 200 dollars invested in Amazon." □

(b)[2 pts] What is $P(a, b, Microsoft)$?

Proof. $P(a, b, Microsoft)$ is "a has b dollars invested in Microsoft." □

(c)[2 pts] Is $Q(a, b)$ a proposition? (just answer "yes" or "no")

Solution. No. □

(d)[4 pts] What is $Q(3, 2)$? Is $Q(3, 2)$ a proposition?

Solution. $Q(3, 2)$ is "$\frac{3}{2} = 1$". This is a (false) proposition. □

(2) Let $P(x, y)$ be "x is friends with y" and let $Q(a, b)$ be "a and b are roommates." Let the domain for all variables be the collection of all UCCS students. Translate the following into English using the method I taught you in the notes/videos.

(a)[3 pts] $\forall x \forall y P(x, y)$

Solution. For every UCCS student x and for every UCCS student y, x is friends with y. □

(b)[3 pts] $\forall x \exists y (P(x, y) \wedge Q(x, y))$

Solution. For every UCCS student x, there exists a UCCS student y such that x is friends with y and x and y are roommates. □

(c)[3 pts] $\exists x \exists y \forall z (P(x, y) \rightarrow (\neg Q(z, x)))$

Solution. There exists a UCCS student x and there exists a UCCS student y such that for every UCCS student z, if x is friends with y, then z and x are not roommates. □

(d)[3 pts] $\exists x \forall y ((\neg P(x, y)) \leftrightarrow (\neg Q(x, y)))$

Solution. There exists a UCCS student x such that for every UCCS student y, x is not friends with y if and only if x are y are not roommates. □

(e)[3 pts]$(\forall x P(John, x) \to \exists y Q(John, y))$

Solution. If for every UCCS student x, John is friends with x, then there exists a UCCS student y such that John and y are roommates. ☐

(3) Let $P(x, y, z)$ be "$x + y = z$" and let $Q(a, b)$ be "$a < b$" ("a is less than b"). Do the following (no justification required):

(a)[3 pts] Find a common domain for x, y, and z (that is, the domain for all variables is the same) consisting of some collection of numbers such that $\forall x \forall y \forall z P(x, y, z)$ is true (hint: can you find a domain containing only one element which makes this true?)

Solution. Let the domain consist solely of the real number 0. ☐

(b)[3 pts] Find a common domain (again, of real numbers) for x, y, and z making $\forall x \forall y \forall z P(x, y, z)$ false.

Solution. Let the domain be the set of all real numbers. ☐

(c)[3 pts] Find a common domain of real numbers for a and b making $\forall a \exists b Q(a, b)$ true.

Solution. Let the domain be the set of all real numbers, as above. ☐

(d)[3 pts] Find a common domain of real numbers for a and b making $\forall a \exists b Q(a, b)$ false.

Proof. Let the domain consists solely of the real number 0, as in (a) above. ☐

(4) Using the notes (see relevant examples I did), justify why the following are formulas:

(a)[5 pts] $(\forall x \exists y P(x, y) \to (Q(z) \vee R(a, b)))$

Solution. $R(a, b)$ and $Q(z)$ are formulas by 1.; by 2., $(Q(z) \vee R(a, b))$ is a formula. $P(x, y)$ is a formula by 1.; by 3., $\exists y P(x, y)$ is a formula; by 3. again, $\forall x \exists y P(x, y)$ is a formula. Finally by 2., $(\forall x \exists y P(x, y) \to (Q(z) \vee R(a, b)))$ is a formula. ☐

(b)[5 pts] $\forall z(R(z) \leftrightarrow \exists x \exists y P(x, y))$

Solution. $P(x, y)$ is a formula by 1.; $\exists y P(x, y)$ is a formula by 3.; $\exists x \exists y P(x, y)$ is a formula by 3.; by 1., $R(z)$ is a formula; by 2., $(R(z) \leftrightarrow \exists x \exists y P(x, y))$ is a formula. Now by 3., $\forall z(R(z) \leftrightarrow \exists x \exists y P(x, y))$ is a formula. ☐

(c)[5 pts] $(\exists x P(x) \wedge \exists y \exists z \forall x(Q(z, x) \vee R(y, x)))$

Proof. $R(y, x)$ and $Q(z, x)$ are formulas by 1.; by 2., $(Q(z, x) \vee R(y.x))$ is a formula; by applying 3. three times, $\exists y \exists z \forall x(Q(z, x) \vee R(y, x))$ is a formula. By 1., $P(x)$ is a formula; by 3., $\exists x P(x)$ is a formula. Finally by 2, $(\exists x P(x) \wedge \exists y \exists z \forall x(Q(z, x) \vee R(y, x)))$ is a formula. ☐

(5) Use the recursive definition of "freeness" to justify the following (see examples in notes). You do NOT need to justify why particular subformulas actually are subformulas, so please don't confuse this with those kinds of problems from (4).

(a)[4 pts] z occurs free in the formula $\exists x \exists y P(x, y, z)$

Solution. z occurs free in $P(x, y, z)$ by 1; z occurs free in $\exists y P(x, y, z)$ by 4.; z occurs free in $\exists x \exists y P(x, y, z)$ by 4. ☐

(b)[4 pts] z occurs free in $(\forall x \exists y P(x, y) \rightarrow (Q(z) \vee R(a, b)))$

Solution. z occurs free in $Q(z)$ by 1.; z occurs free in $(Q(z) \vee R(a, b))$ by 3.; by 3. again, z occurs free in $(\forall x \exists y P(x, y) \rightarrow (Q(z) \vee R(a, b)))$. ☐

(c)[4 pts] y occurs free in $\forall z(R(z) \leftrightarrow \exists x P(x, y))$

Solution. y occurs free in $P(x, y)$ by 1.; y occurs free in $\exists x P(x, y)$ by 4.; y occurs free in $(R(z) \leftrightarrow \exists x P(x, y))$ by 3.; finally, y occurs free in $\forall z(R(z) \leftrightarrow \exists x P(x, y))$ by 5. ☐

(d)[4 pts] y occurs free in $(\exists x P(y) \wedge \exists y \exists z \forall x(Q(z, x) \vee R(y, x)))$

Solution. y occurs free in $P(y)$ by 1.; by 4., y occurs free in $\exists x P(y)$. Finally by 3., y occurs free in $(\exists x P(y) \wedge \exists y \exists z \forall x(Q(z, x) \vee R(y, x)))$. ☐

(6) Let $C(x, y)$ be "x and y have chatted online" and let $I(x)$ be "x has an internet connection." Translate the following into formulas, and be sure to use correct syntax. You can take the domain for x and y to be the collection of all people, though the domain really is irrelevant and you can do the following even without a domain being specified.

(a)[4 pts] Every person that has an internet connection has chatted with someone online.

Solution. $\forall x(I(x) \rightarrow \exists y C(x, y))$. ☐

(b)[4 pts] John has chatted with everyone online (your formula should have "John" in it; see problem (1)).

Solution. $\forall y C(John, y)$. ☐

(c)[4 pts] No one has an internet connection.

Solution. $(\neg \exists x I(x))$. ☐

(d)[4 pts] If Sally has chatted with someone online, then Sally has an internet connection.

Solution. $(\exists y C(Sally, y) \rightarrow I(Sally))$. ☐