Math 2150 Assignment 5 Solutions [each part of each problem is worth 5 pts]

(1) Negate so that negation symbols immediately precede propositional variables. Be careful to use correct syntax (1. - 5. of the June 28 notes will be of use here). Just giving the formulas is sufficient here; you do not need to show work.

(a) $((\neg q) \to (\neg p))$

Solution. The negation of "if a, then b" is "a and not b", and so the initial negation is "$(\neg q)$ and not $(\neg p)$", that is, "$(\neg q)$ and p". Thus the final formula is $((\neg q) \wedge p)$. $\square$

(b) $(r \wedge (s \leftrightarrow t))$

Solution. By DeMorgan, the negation of "a and b" is "not a or not b". Thus the initial negation is "not r or not $(s \leftrightarrow t)$". The first "not" immediately precedes a variable, but the second does not. Look at the notes: how do we negate a formula of the form $(a \leftrightarrow b)$? It is $((a \wedge (\neg b)) \vee ((\neg a) \wedge b))$. Thus the desired negation is $((\neg r) \vee ((s \wedge (\neg t)) \vee ((\neg s) \wedge t)))$. $\square$

(c) $(p \vee (\neg q))$

Solution. By Demorgan, the negation of "a or b" is "not a and not b". Thus the initial negation is "not p and not $(\neg q)$", which becomes "not p and q". Thus the final formula is $((\neg p) \wedge q)$. $\square$

(d) $((p \vee q) \wedge r)$

Solution. Again, by Demorgan, the negation of "a and b" is "not a or not b". Thus the initial negation is "not $(p \vee q)$ or not r". Applying DeMorgan to negate $(p \vee q)$, we get $((\neg p) \wedge (\neg q))$. Thus the final negation is $(((\neg p) \wedge (\neg q)) \vee (\neg r))$. $\square$

(2) Negate so that negation symbols immediately precede predicates (see 1. - 5. of June 28 notes along with Theorem 1). Again, you need not show work here (though doing so may help increase partial credit).

(a) $\forall x(P(x) \vee Q(x))$

Solution. From DeMorgan, the negation of $\forall x\alpha$ is $\exists x(\neg\alpha)$. So the initial negation is "exists x such that not $(P(x) \vee Q(x))$". To negate the final formula, first note that there are no quantifiers after the "not", and so we are back in propositional logic. So the negation from propositional DeMorgan is $((\neg P(x)) \wedge (\neg Q(x)))$. So the final negated formula is $\exists x((\neg P(x)) \wedge (\neg Q(x)))$. $\square$

(b) $\exists x(P(x) \to (Q(x) \vee R(x)))$

Solution. From DeMorgan, the negation of $\exists x\alpha$ is $\forall x(\neg\alpha)$. So the initial negation is "for all x, not $(P(x) \to (Q(x) \vee R(x)))$". Again, we now find ourselves in propositional logic following the "not". The negation of "if A, then B" is (from above, and I've yelled about this MANY times as well as put this in all caps in the notes) "A and not B", which becomes "$P(x)$ and not $(Q(x) \vee R(x))$"; using DeMorgan again, "$P(x)$ and not $(Q(x) \vee R(x))$" becomes "$P(x)$ and (not $Q(x)$ and not $R(x)$)". So the final answer is the following: $\forall x(P(x) \wedge ((\neg Q(x)) \wedge (\neg R(x))))$. $\square$

(c) $\exists x(P(x) \to \forall y \forall z Q(y, z))$

Solution. From Demorgan, the negation of $\exists x\alpha$ is $\forall x(\neg\alpha)$. Thus from above, the negation of $\exists x(A \to B)$ is $\forall x(A \wedge (\neg B))$. So our initial negation is "$\forall x(P(x)$ and not $\forall y \forall z Q(y, z))$". Now apply Demorgan twice to negate $\forall y \forall z Q(y, z)$ to get $\exists y \exists z(\neg Q(y, z))$. Thus the final negation is $\forall x(P(x) \wedge \exists y \exists z(\neg Q(y, z)))$. $\square$

(d) $\exists x \forall y \exists z(P(x, y) \wedge Q(z))$

Solution. Putting the above ideas together (begin by using predicate DeMorgan three times), the negation is $\forall x \exists y \forall z((\neg P(x, y)) \vee (\neg Q(z)))$. $\square$

(3) Show that the following formulas are not logically equivalent:

(a) $\exists x(P(x) \wedge Q(x))$ and $(\exists x P(x) \wedge \exists x Q(x))$

Solution. Let the domain for x be the collection of all (U.S.) coins. Let $P(x)$ be "x is a penny" and let $Q(x)$ be "x is a quarter". The first formula above translates to "There exists a coin x such that x is a penny and x is a quarter." But no coin is both at the same time, so the first formula is false relative to this domain. But the second translates to "There exists a coin x such that x is a penny or there exists a coin x such that x is a quarter." This formula is true. This proves that the two formulas are not logically equivalent. $\square$

(b) $\forall x(P(x) \vee Q(x))$ and $\forall x(P(x) \wedge Q(x))$

Solution. We can adjust the above domain just slightly but use the same predicates. Let the domain for x be the collection of all coins which are either pennies or quarters. Then the first formula is true but the second is clearly false. $\square$

(c) $\forall x(P(x) \to Q(x))$ and $\forall x(P(x) \leftrightarrow Q(x))$

Solution. Let the domain for x be the collection of all real numbers. Let $P(x)$ be "x is an odd integer" and let $Q(x)$ be "x is an integer". Then the first formula is true but the second is false, since there are integers which are not odd. $\square$

(4) Let $T(x)$ be "x is tall", let $F(x, y)$ be "x and y are friends", and let $S(x, z)$ be "x plays z". Let the domains for x and y be the collection of all people and the domain for z be the collection of all sports. NOTE: there is nothing "magical" about the letters chosen above. For example (see previous hw), $S(a, b)$ is "a plays b". As another example, "$F(x, y) \wedge S(y, golf)$" is "$x$ and y are friends and y plays golf."

(a)(i) Translate "Everyone has a friend that plays basketball" into a formula.

Solution. $\forall x \exists y(F(x, y) \wedge S(y, basketball))$. □

(a)(ii) Negate your formula in (i) so that all negation symbols immediately precede predicates.

Solution. $\exists x \forall y((\neg F(x, y)) \vee (\neg S(y, basketball)))$. □

(a) (iii) Translate your formula from (ii) into English.

Solution. There exists a person x such that for every person y, x and y are not friends or y does not play basketball. □

(b)(i) Translate "There exists a tall person that has no friends" into a formula.

Solution. $\exists x(T(x) \wedge \forall y(\neg F(x, y)))$. □

(b)(ii) Negate your formula in (i) so that all negation symbols immediately precede predicates.

Solution. $\forall x((\neg T(x)) \vee \exists y F(x, y))$. □

(b)(iii) Translate your formula from (ii) into English.

Solution. For every person x, either x is not tall or there exists a person y such that x and y are friends. □