Chapter 3

The z-Transform

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3.1 Introduction

A generalization to the Fourier transform of a sequence is the z-transform. In the continuous-time the corresponding generalization is the Laplace transform. The z-transform has the following advantages over the Fourier transform:

- Converges for a broader class of signals
- Analytically provides a more convenient notation
- Allows the power of complex variable theory to be effectively utilized

3.2 The Bilateral z-Transform

The z-transform of a sequence is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z is a complex variable.

- A convenient operator notation which we will adopt is to write

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z)$$

- Note that the z-transform operator transforms the sequence $x[n]$ to $X(z)$, a function of a continuous complex variable z

- The relationship between a sequence and its transform is denoted as

$$x[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$$

- The z-transform we have defined above is actually the *two-sided* or *bilateral z-transform*, which in general is different from the *one-sided* or *unilateral z-transform*, which is defined as

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Fourier Transform Connection

- An important comparison between the z-transform and the Fourier transform can be made by first writing z in polar form

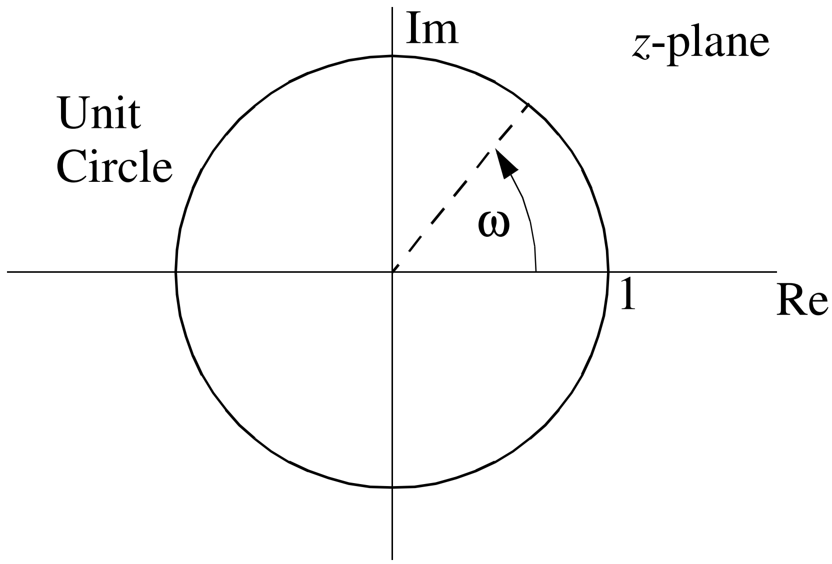
$$z = r\,e^{j\omega}$$

then

$$\begin{aligned} X(re^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{\infty} (x[n]\,r^{-n})e^{-j\omega n} \end{aligned}$$

- The above representation of $\mathcal{Z}\{x[n]\}$ can also be viewed as $\mathcal{F}\{x[n]r^{-n}\}$, where the special case $r = 1$ gives the Fourier transform of $x[n]$

- Considering $X(z)$ as a function defined over the complex plane and $z = e^{j\omega}$ the unit circle in the z-plane, we see that the Fourier transform is just the z-transform evaluated on the unit circle



Complex z-plane showing the unit circle

- Since ω is the angle a vector with point on the unit circle makes to the real axis, it should be clear that $X(e^{j\omega})$ does indeed have period 2π (Note: $\omega = 0 \leftrightarrow z = 1$, $\omega = \pi/2 \leftrightarrow z = j$, $\omega = -\pi \leftrightarrow z = -1$, and $\omega = -\pi/2 \leftrightarrow z = -j$)

3.2.1 Convergence Considerations

We know that the Fourier transform does not converge for all sequences, similarly the z-transform does not converge for all sequences nor does it in general converge over the entire z-plane.

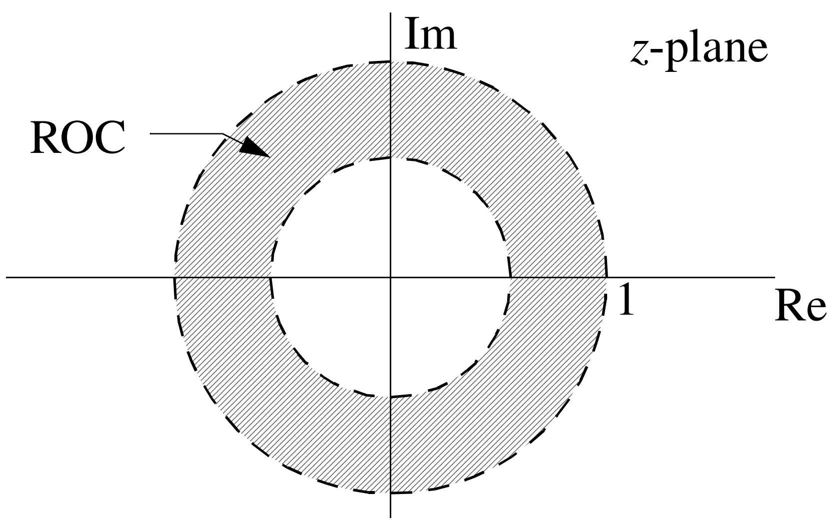
- Define: the values in the z-plane for which the z-transform converges as the *region of convergence* or ROC
- If we extend the uniform convergence requirement of the Fourier transform to the z-transform, we then have that the z-transform must also be absolutely summable

- The z-transform is absolutely summable if

$$\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n}$$
$$= \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

- Note that convergence depends only on $|z|$, thus if the series converges for $z = z_1$, the ROC also contains the circle $|z| = |z_1|$

- In general the region of convergence is a ring or annulus in the z-plane



General ROC

- An important observation is to note that if the unit circle is contained in the ROC, then the z-transform converges for $|z| = 1$ and hence the Fourier transform also converges

- From complex variable theory we know that the z-transform is a *Laurent series*
  - The function corresponding to a Laurent series within the region of convergence is also analytic
  - Hence the z-transform and all of its derivatives are continuous functions of z within the ROC
  - In particular if the ROC includes the unit circle, then the Fourier transform and all of its derivatives with respect to ω are continuous functions of ω. Additionally since the sequence must be absolutely summable, it also is stable
- It is important to note that certain sequences such as $\sin(\omega_c n)/(\pi n)$ and $\cos\omega_o n$ were not absolutely summable in the Fourier transform case, yet under different convergence criteria or by allowing impulses, the Fourier transforms could be obtained
- For special cases such as this the Fourier transforms are not continuous and hence **do not** result from evaluating the z-transform on the unit circle
- Frequently the z-transforms of interest to us will converge to a ratio of polynomials in z inside the ROC, i.e. $X(z) = P(z)/Q(z)$
- In this case we will see that the *poles* of $X(z)$, i.e. values of z where $Q(z) = 0$) will determine the ROC. Note the *zeros* of $X(z)$ are the values of z where $P(z) = 0$.

Example 3.1: A Right-Sided Sequence

Find the z-transform of $x[n] = a^n u[n]$ and the corresponding ROC

- According to the definition we have

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- Convergence requires that

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty \quad \Rightarrow \quad |a/z| < 1 \text{ or } |z| > |a|$$

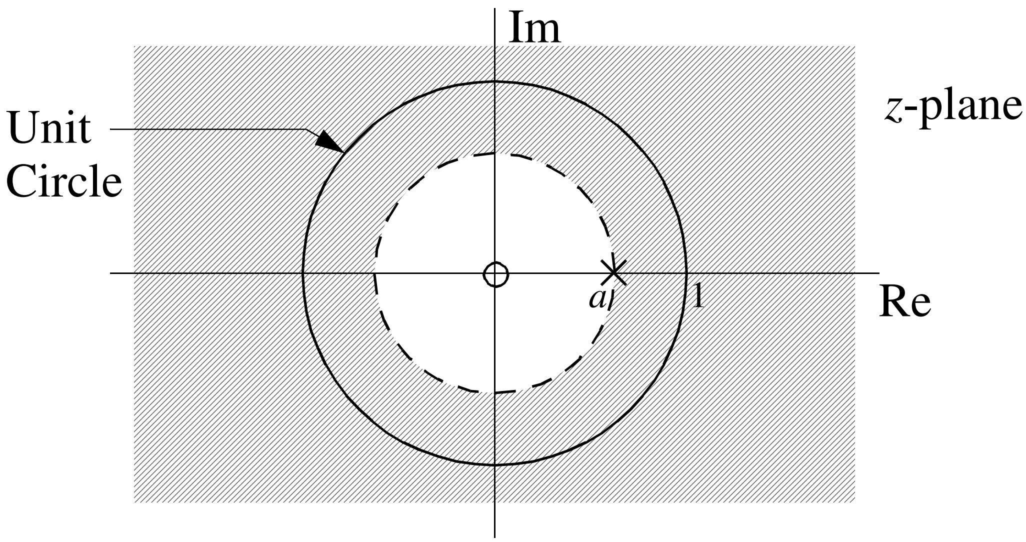
thus the ROC = $\{z : |z| > |a|\}$

- For z inside the ROC we can write

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$

- Suppose $a = 1$, then $x[n]$ is the unit step function and we see that the ROC is $|z| > 1$

- When the z-transform of a sequence is a rational function it is instructive to plot the pole-zero locations in the z-plane along with the region of convergence



Pole-zero plot and ROC for $a^n u[n]$

- In the above pole-zero plot we have a zero at $z = 0$ and a pole at $z = a$ which sets the inner radius of the ROC

Example 3.2: A Left-Sided Sequence

A related signal is the time reversed exponential given by $x[n] = -a^n u[-n-1]$

- The z-transform is

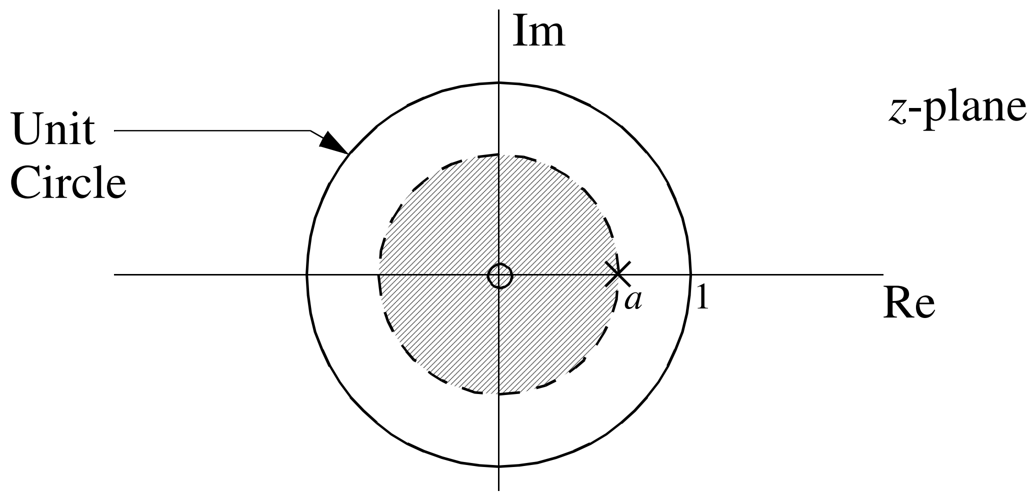
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\underbrace{\sum_{n=-\infty}^{-1}}_{\text{reindex}} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

- The series converges provided $|z/a| < 1$ or $|z| < |a|$, thus

$$X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}, \quad |z| < |a|$$

- Note that both exponential sequences have the same $X(z)$, however the ROCs are different



Pole-zero plot and ROC for $-a^n u[-n-1]$

<u>*Comment*</u>

The two examples given above point out the importance of indicating the ROC when dealing with the z-transform of a sequence. Clearly the relationship between $x[n]$ and $X(z)$ is not unique without knowledge of the ROC.

Example 3.3: Two Right-Sided Exponential Sequences

Find the z-transform and ROC of the sequence

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]$$

- To simplify the computation we note that $x[n]$ is composed of two exponential sequences which we know have transform pair

$$a^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

- Further note that the z-transform operator is linear, i.e.

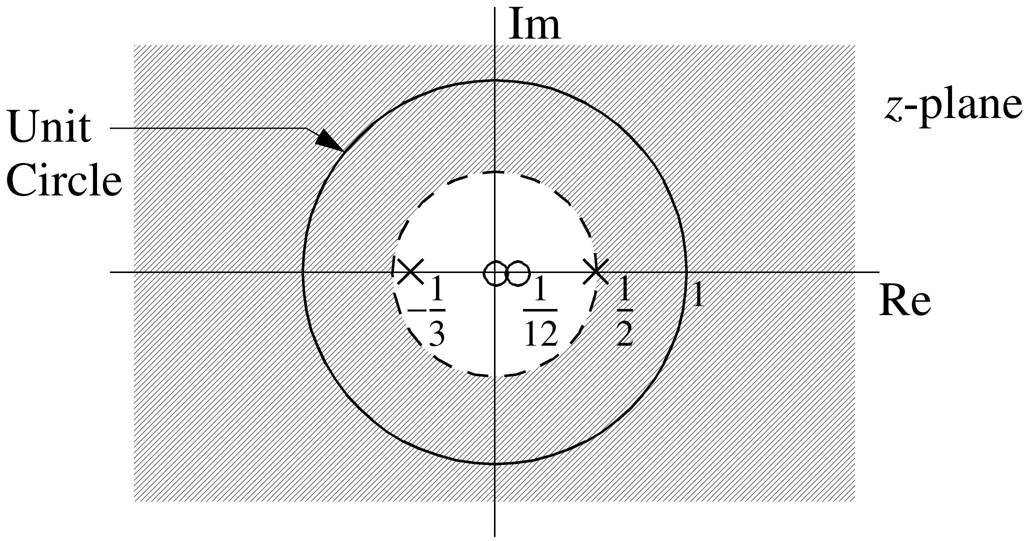
$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z), \quad z \in \text{ROC}$$

- The ROC must contain values of z such that **both** series converge, hence ROC $=$ ROC$_1 \cap$ ROC$_2$ (i.e. the intersection of the individual ROCs)

- Returning to the example we have

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}} = \frac{2z(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

with the ROC requiring that $|z| > 1/2$ and $|z| > 1/3$, thus the ROC $= |z| > \max\{1/2, 1/3\} = 1/2$



Pole-zero plot and ROC

Example 3.4: A Left and Right-Sided Exponential Sequence

Rework the previous example with the first exponential time reversed, that is let

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1] + \left(-\frac{1}{3}\right)^n u[n]$$

- Using the results from the previous examples we know that

$$\left(-\frac{1}{3}\right)^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}$$

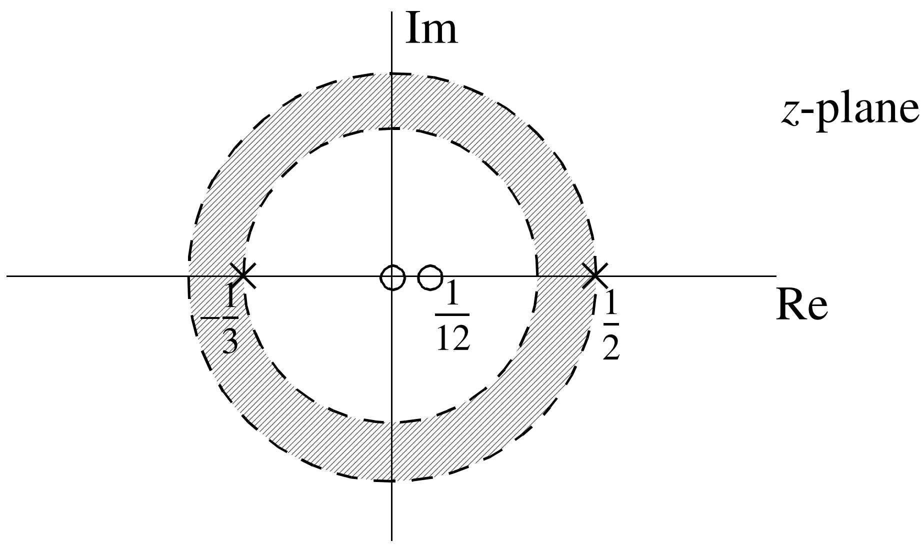
and

$$-\left(\frac{1}{2}\right)^n u[-n-1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

- Using linearity we can immediately write that

$$X(z) = \frac{2(1 - \frac{1}{12}z^{-1})}{(1 + \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{2z(z - \frac{1}{12})}{(z + \frac{1}{3})(z - \frac{1}{2})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$

- Note that although $X(z)$ has the same functional form as in the previous example, the ROC is now an annular region



Pole-zero plot and ROC

3.2.2 Convergence for Finite Length Sequences

If $x[n]$ has finite length then $X(z)$ converges everywhere so long as each term $|x[n]z^{-n}|$ is finite.

- For $x[n]$ nonzero only on $N_1 \le n \le N_2$

$$X(z) = x[N_1]z^{-N_1} + \cdots + x[N_2]z^{-N_2}$$

- If $X(z)$ includes only positive powers of z, then the ROC is the entire z-plane except for $|z| \to \infty$, (i.e. ROC $= |z| < \infty$)
- If $X(z)$ includes only negative powers of z, then the ROC $= |z| > 0$
- If $X(z)$ contains both positive and negative powers of z, then the ROC $= 0 < |z| < \infty$

Example 3.5: **Rectangular Window**

An important finite length sequence is the rectangular window of length N with exponential weighting a^n,

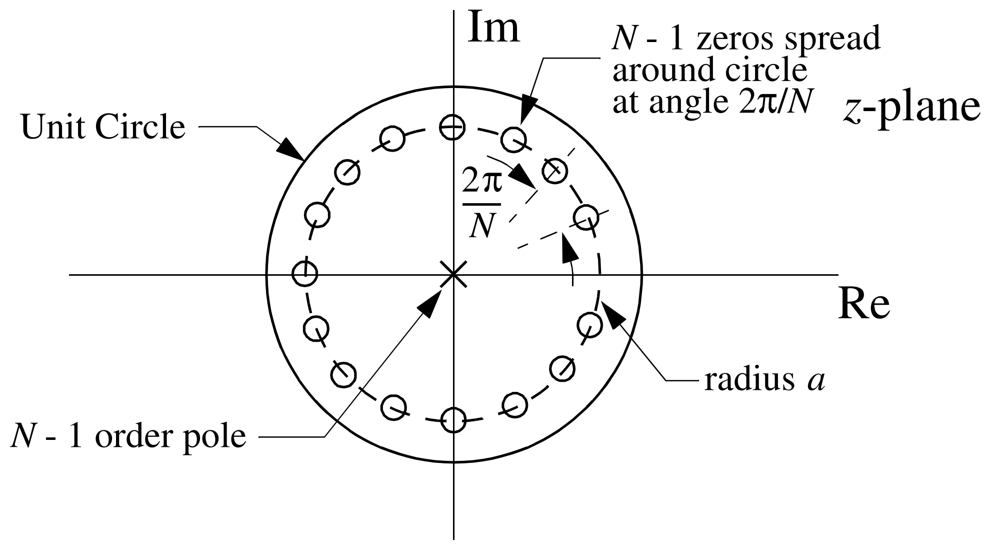
$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

- By definition

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a} \end{aligned}$$

- Since $X(z)$ includes only negative powers of z, the ROC $= |z| > 0$
- From the summed form of $X(z)$ we see that the pole-zero pattern consists of $N-1$ zeros uniformly spaced about zero with radius a, excluding the zero at $z = a$, since it is cancelled by a pole at $z = a$ (Note the N roots of $z^N - a^N = 0$ are $z_k = ae^{j(2\pi k/N)}, \;\; k = 0, 1, \ldots, \; N-1$)

- There are also $N-1$ poles at zero which is why the origin is excluded from the ROC



Pole-zero plot for $N = 16$ and $a < 1$

3.3 Properties of the ROC

The following properties assume that the z-transform is a rational function and that $|x[n]|$ is finite except perhaps at $n \to \infty$ or $n \to -\infty$.

Property 1: The ROC is an annulus in the z-plane centered at the origin, i.e. ROC $= \{z : 0 \leq r_R < |z| < r_L \leq \infty\}$ where r_R and r_L are the inner and outer radii of the annulus respectively.

Property 2: The Fourier transform of sequence $x[n]$ converges absolutely if and only if the ROC of $X(z)$ includes the unit circle.

- Follows from the fact that the z-transform reduces to the Fourier transform when $|z| = 1$

Property 3: The ROC **cannot** contain any poles.

- This follows from the fact that $X(z)$ is infinite at a pole and hence the series does not converge

Property 4: For $x[n]$ a finite duration sequence the ROC is the entire z-plane except possibly $z = 0$ or $z \to \infty$.

- Follows from the definition of the z-transform

Property 5: If $x[n]$ is a *right-sided sequence*, that is $x[n]$ is zero for $n < N_1 < \infty$, then the ROC is the **exterior** of a circle with radius equal to the magnitude of the largest pole of $X(z)$. Note $z \to \infty$ may be excluded from the ROC depending upon the sign of N_1.

- To show that the ROC is the exterior of a circle suppose that the ROC includes $|z| = r_0$, then we know that $x[n]r_0^{-n}$ is absolutely summable

- The sequence $x[n](r_0 + \Delta r)^{-n}$, $\Delta r > 0$, is also absolutely summable since $(r_0 + \Delta r)^{-n}$ will decay faster than r_0^{-n} for positive values of n, and there are only a finite number of negative n terms since $x[n]$ is right-sided

- The fact that the circle radius corresponds to the magnitude of the largest pole is easiest to show if we assume that all the poles are simple

- If we expand $X(z)$ using partial fractions, then each term in the expansion will correspond to a simple pole and have ROC corresponding to the exterior of a circle with radius equal to the particular pole magnitude (recall that a single pole results from an exponential sequence)

- The ROC of a sum of single pole terms corresponds to the intersection of the ROCs, thus since all the ROCs are the exteriors of circles, it is the pole with largest magnitude that determines the ROC

Property 6: If $x[n]$ is a *left-sided sequence*, that is $x[n]$ is zero for $n > N_2 > -\infty$, then the ROC is the **interior** of a circle with radius equal to the magnitude of the smallest pole of $X(z)$. Note $z = 0$ may be excluded from the ROC depending upon the sign of N_2.

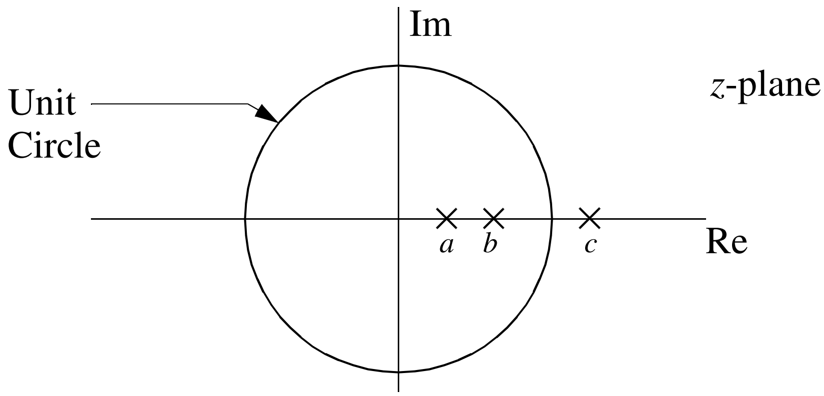
- The justification of property 6 follows the same logic that was used in justifying property 5

Property 7: If $x[n]$ is a *two-sided sequence*, that is the sequence is infinite in duration in both directions, then the ROC is an annulus with inner and outer radius bounds determined by the poles of $X(z)$. Specifically for two-sided sequences some poles correspond to right-sided sequence components, while others correspond to left-sided sequence components. The inner radius, r_R, is given by the right-sided sequence pole with largest magnitude and the outer radius, r_L, is given by the left-sided sequence pole with smallest magnitude. The complete ROC is then the intersection of the right-sided and left-sided sequence ROCs.

Property 8: The ROC must be a connected region.

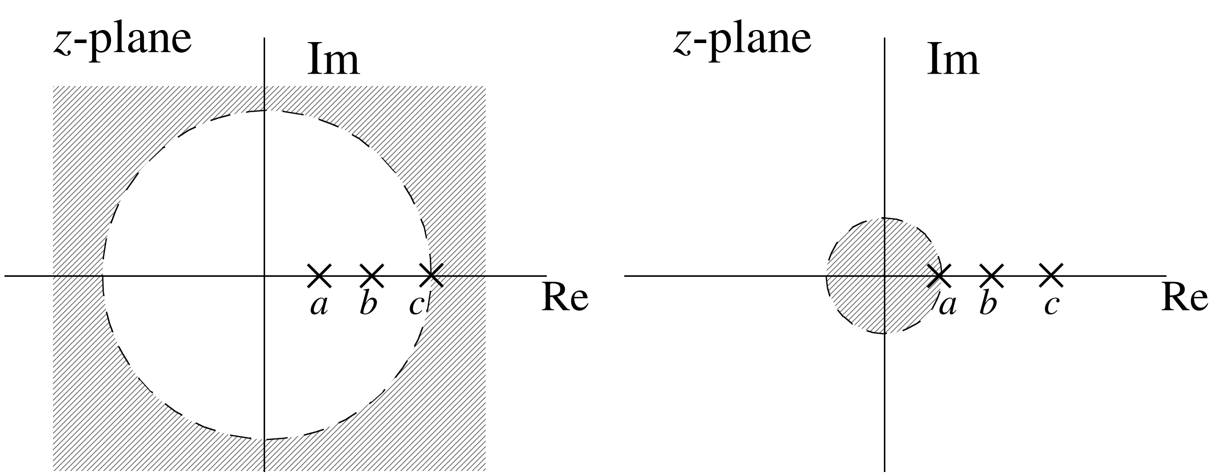
Example 3.6: A Three Pole Signal

Consider the pole-zero pattern shown below

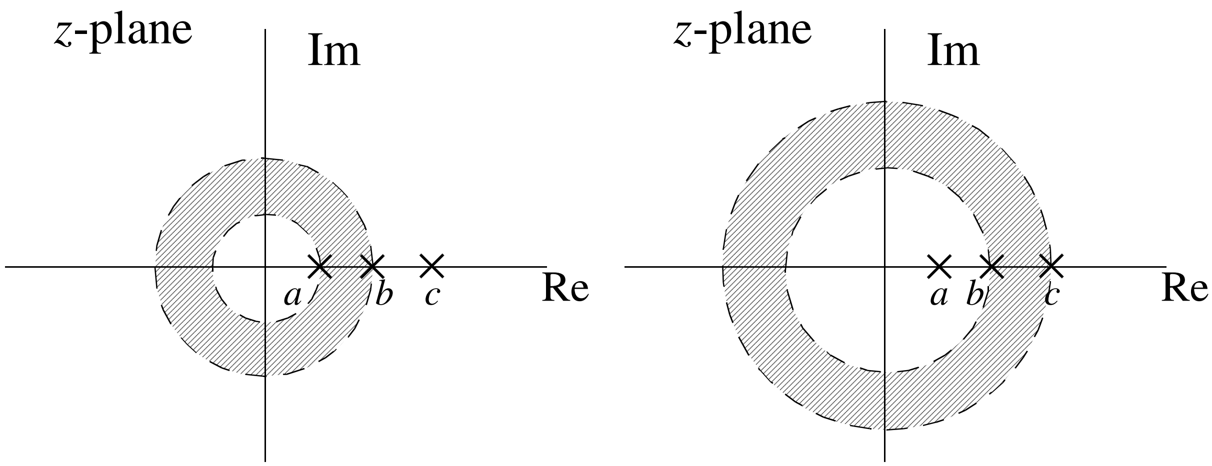


A three pole $X(z)$

- If $X(z)$ has three poles, then there are four possible choices for the ROC



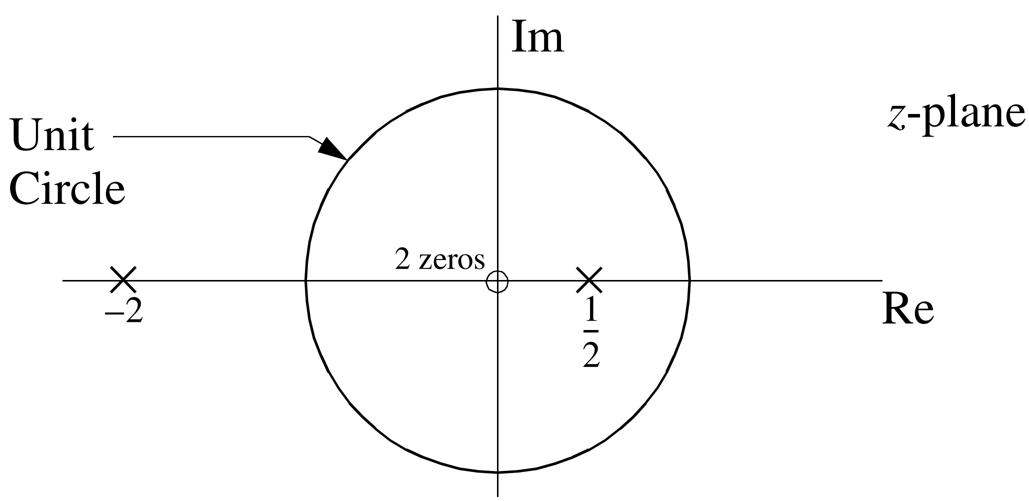
Case (a) $\{z : |z| > c\}$ and (b) $\{z : |z| < a\}$



Case (c) $\{z : a < |z| < b\}$ and (d) $\{z : b < |z| < c\}$

Example 3.7: **Stability, Causality, and the ROC**

- Consider an LTI system having a z-transform with the following pole-zero plot:

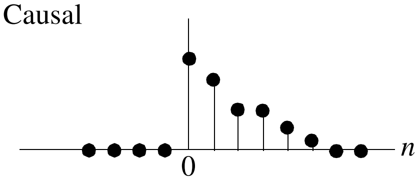
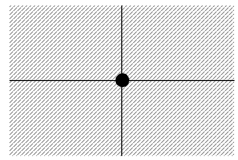
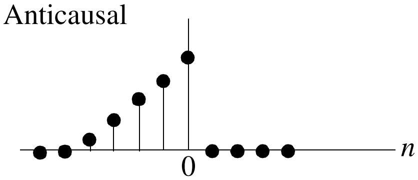
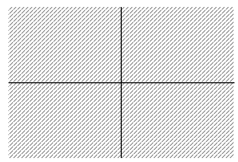
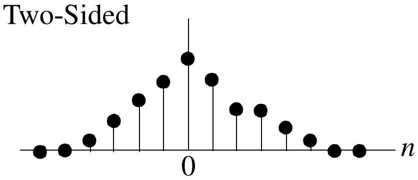
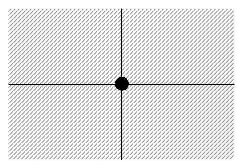


Pole-zero plot of $H(z)$

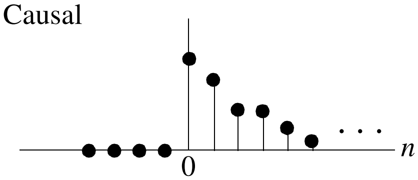
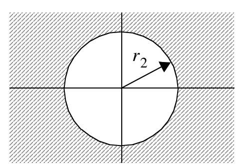
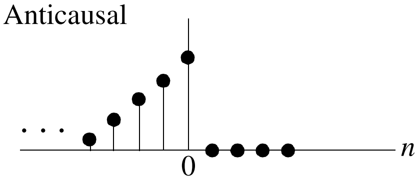
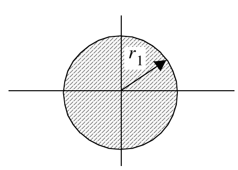
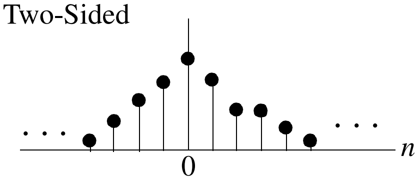
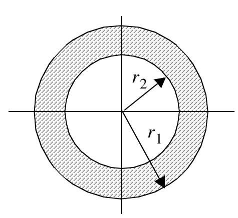
- There are three valid ROCs for this system

- If the system is known to be **stable** ($h[n]$ absolutely summable), then the ROC must include the unit circle, so ROC: $1/2 < |z| < 2$
  - Under this condition the system is not causal since we have right and left sided terms
- On the other hand if it is known that the system is **causal**, then it must be that the ROC is the exterior of the circle corresponding to the largest pole radius, ROC: $|z| > 2$
  - Now it is clear that the system can no longer be stable because of a pole contributing a right sided sequence of the form $(-2)^n u[n]$
- In summary, for this system it is not possible to be both stable and causal

3.3.1 ROC Summary for Finite-Duration Signals

Signal	ROC	
Causal 		Entire z-plane except $z = 0$
Anticausal 		Entire z-plane except $z = \infty$
Two-Sided 		Entire z-plane except $z = 0$ and $z = \infty$

3.3.2 ROC Summary for Infinite-Duration Signals

Signal	ROC	
Causal 		$\|z\| > r_2$
Anticausal 		$\|z\| < r_1$
Two-Sided 		$r_2 < \|z\| < r_1$

3.4 The Inverse z-Transform

Eventually we will be analyzing LTI systems in the z-domain. Ultimately we may wish to compute the inverse z-transform that results from some algebraic manipulation of z-transforms.

Formally the inverse z-transform is defined as the contour integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}\, dz$$

where the contour C encircles the origin and is chosen to lie inside the ROC. We can evaluate this contour integral using the *Cauchy integral theorem*.

Formal evaluation of the contour integral has traditionally been a part of this course. Learning the contour integration approach is not absolutely essential, however. The second and third editions of the O&S text omits this topic in favor of other newer DSP topics added elsewhere in the text. For historical reasons inverse z-transformation using contour integration is still found in this note set, including a short overview of the needed complex variable theory.

Inversion techniques that are less formal, but get the job done, will be the focus in this course. The techniques or procedures are: *the inspection method*, *partial fraction expansions*, and *power series expansion*.

3.4.1 Inspection Method

With this technique we perform the inverse transform "by inspection" with the aid of a table of transform pairs

A Short Table of z-Transform Pairs

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	all z
$\delta[n-m]$	z^{-m}	all z except $z=0$ if $m>0$ and $\|z\|=\infty$ if $m<0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
$\frac{(n+1)(n+2)\cdots(n+k-1)}{(k-1)!}a^n u[n]$	$\frac{1}{(1-az^{-1})^k}$	$\|z\|>\|a\|$
$r^n\cos(\omega_o n)u[n]$	$\frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\|>r$
$r^n\sin(\omega_o n)u[n]$	$\frac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\|>r$
$\cosh(nb)u[n]$	$\frac{1-(\cosh b)z^{-1}}{1-(2\cosh b)z^{-1}+z^{-2}}$	$\|z\|>\max\{\|e^b\|,\|e^{-b}\|\}$
$\sinh(nb)u[n]$	$\frac{(\sinh b)z^{-1}}{1-(2\cosh b)z^{-1}+z^{-2}}$	$\|z\|>\max\{\|e^b\|,\|e^{-b}\|\}$
$\begin{cases} a^n, & 0\le n\le N-1 \\ 0, & \text{otherwise}\end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\|z\|>0$

3.4.2 Partial Fraction Method

If $X(z)$ cannot be found directly from a table entry, we may be able to rewrite it as a collection of table entries using a partial fraction expansion.

To begin with assume that $X(z)$ is a rational function, then we can write

$$\begin{aligned} X(z) &= \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \\ &= \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} \end{aligned}$$

- As written above $X(z)$ has N poles and M zeros at nonzero locations in the z-plane
- If $M>N$ there will also be $M-N$ poles at $z=0$

- If $N > M$ there will be $N - M$ zeros at $z = 0$
- In all cases the total number of poles equals the total number of zeros
- $X(z)$ is a *proper* rational function if $a_N \neq 0$ and $M < N$
- If $M \geq N$ then $X(z)$ is an improper rational function, which by using long division of the numerator by the denominator can be written as the sum of a polynomial and another proper rational function i.e.,

$$X(z) = \frac{N(z)}{D(z)} = \sum_{r=0}^{M-N} B_r z^{-r} + \frac{N_1(z)}{D(z)}$$

  where $N_1(z)$ is the remainder polynomial having degree less than N

Distinct Poles

- If $X(z)$ is a proper rational function and $D(z)$ contains no repeated roots, then we can write $X(z)$ as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

  where the d_k's are the nonzero poles of $X(z)$

- Note: To avoid confusion with the text we will expand $X(z)$ as shown above using negative powers of z, however in many texts the expansion is done using positive powers of z, then the transform pair tables are modified accordingly to contain factors of the form $z - d_k$

- For the distinct poles case the coefficients, A_k, are given by

$$A_k = (1 - d_k z^{-1}) X(z)|_{z^{-1}=d_k^{-1}}$$

- If $M \geq N$ then the expansion is performed on the proper rational function $N_1(z)/D(z)$ so that

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Multiple-Order Poles

- Suppose that $X(z)$ has a pole of multiplicity s at $z = d_i$, then the general form of the partial fraction expansion is

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

- The coefficients A_k and B_r are obtained as before. The C_m's are obtained from

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \cdot \left\{ \frac{d^{s-m}}{dw^{s-m}} [(1 - d_i w)^s X(w^{-1})] \right\}_{w=d_i^{-1}}$$

- If $X(z)$ contains several multiple-order poles, then the partial fraction expansion will contain a term of the form

$$\sum_{m_i=1}^{s_i} \frac{C_{mi}}{(1 - d_i z^{-1})^{m_i}}$$

for each multiple-order pole

General Inversion Procedure

- Suppose initially that $X(z)$ contains only first order poles, then for the polynomial terms we use

$$B_r\delta[n-r] \stackrel{\mathcal{Z}}{\longleftrightarrow} B_r z^{-r}$$

- For the $A_k/(1-d_k z^{-1})$ terms we need to consider the general form of the ROC: $r_R < |z| < r_L$
  - If the pole $z = d_k$ has $|d_k| \leq r_R$ then the corresponding sequence is right-sided and we inverse transform using

$$A_k d_k^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}}$$

  - If the pole $z = d_k$ has $|d_k| \geq r_L$ then the corresponding sequence is left-sided and we inverse transform using

$$-A_k d_k^n u[-n-1] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{A_k}{1-d_k z^{-1}}$$

- Multiple-order poles are divided into left-sided and right-sided sequence contributions in the same way

Example 3.8:

Consider a sequence $x[n]$ with z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}, \quad \text{ROC} = |z| > 1$$

- To begin with note that $M = N = 2$ which implies that before performing the partial fraction expansion we must perform long division to reduce the remainder poly $N_1(z)$ to degree 1, i.e.

$$\begin{array}{r|l} & \quad\quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ \hline & \quad\quad 5z^{-1} - 1 \end{array}$$

The division is stopped when the order of the remainder is $z^{-(N-1)} = z^{-1}$.

- We can now write

$$X(z) = 2 + \frac{5z^{-1} - 1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

- Next the denominator poly is factored into two real roots, i.e.

$$D(z) = \left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})$$

- The partial fraction expansion is

$$X(z) = 2 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

- Solve for A_1 and A_2

$$\begin{aligned} A_1 &= \left.\frac{-1+5z^{-1}}{1-z^{-1}}\right|_{z^{-1}=2} = \frac{-1+10}{1-2} = -9 \\ A_2 &= \left.\frac{-1+5z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=1} = \frac{-1+5}{1-\frac{1}{2}} = 8 \end{aligned}$$

- Finally

$$X(z) = 2 - \frac{9}{1-\frac{1}{2}z^{-1}} + \frac{8}{1-z^{-1}}$$

- Since both poles have magnitude less than or equal to one, the exponential sequences are right-sided, thus

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

Example 3.9:

Since the $X(z)$ in the previous example contains two distinct poles there are a total of three valid ROC definitions. The two remaining cases are:

$$\begin{aligned} &\text{case 2:} && \text{ROC} = |z| < \tfrac{1}{2} \\ &\Rightarrow && x[n] = 2\delta[n] + 9\left(\tfrac{1}{2}\right)^n u[-n-1] - 8u[-n-1] \\ &\text{case 3:} && \text{ROC} = \tfrac{1}{2} < |z| < 1 \\ &\Rightarrow && x[n] = 2\delta[n] - 9\left(\tfrac{1}{2}\right)^n u[n] - 8u[-n-1] \end{aligned}$$

Example 3.10:

Find the inverse z-transform of

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}, \quad \text{ROC} = |z| > 1$$

- Observe that $X(z)$ has a simple pole at $z = -1$ and a repeated pole at $z = 1$, thus the partial fraction expansion is of the form

$$X(z) = \frac{A_1}{1+z^{-1}} + \frac{C_1}{1-z^{-1}} + \frac{C_2}{(1-z^{-1})^2}$$

- We can immediately solve for A_1

$$A_1 = \left.\frac{1}{(1-z^{-1})^2}\right|_{z^{-1}=-1} = \frac{1}{2^2} = \frac{1}{4}$$

- Using the multiple-pole coefficients formula

$$\begin{aligned} C_1 &= \frac{1}{1!(-1)^1}\left\{\frac{d}{d\,w}\left(\frac{1}{1+w}\right)\right\}_{w=1} = \frac{1}{4} \\ C_2 &= \frac{1}{0!(-1)^0}\left\{\frac{1}{1+w}\right\}_{w=1} = \frac{1}{2} \end{aligned}$$

- Thus we can write

$$X(z) = \frac{1/4}{1+z^{-1}} + \frac{1/4}{1-z^{-1}} + \frac{1/2}{(1-z^{-1})^2}$$

- To invert term-by-term we note that each term is associated with a right-sided exponential sequence and use the fact that

$$(n+1)a^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1-az^{-1})^2}$$

Hence,

$$x[n] = \frac{1}{4}(-1)^n u[n] + \frac{1}{4}u[n] + \frac{1}{2}(n+1)u[n]$$

Inversion of Distinct Complex Conjugate Pole Pairs

Consider the partial fraction expansion of a general complex conjugate pole pair ($N(z)$ is of degree one and assumed to have real coefficients)

$$\begin{aligned} X_k(z) &= \frac{b_0 + b_1 z^{-1}}{(1-d_k z^{-1})(1-d_k^* z^{-1})} \\ &= \frac{A_k}{1-d_k z^{-1}} + \frac{A_{kk}}{1-d_k^* z^{-1}} \end{aligned}$$

- Solving for the coefficients we have

$$\begin{aligned} A_k &= \frac{b_0 + b_1/d_k}{1-d_k^*/d_k} \\ A_{kk} &= \frac{b_0 + b_1/d_k^*}{1-d_k/d_k^*} = A_k^* \end{aligned}$$

- Thus we see that the coefficients in the expansion are also conjugates

- Assuming the ROC is $|z| > |d_k|$ the inverse transform is the right-sided sequence

$$\begin{aligned} x_k[n] &= \left[A_k(d_k)^n + A_k^*(d_k^*)^n\right] u[n] \\ &= |A_k| r_k^n \left[e^{j(\beta_k n + \alpha_k)} + e^{-j(\beta_k n + \alpha_k)}\right] u[n] \\ &= 2|A_k| r_k^n \cos(\beta_k n + \alpha_k) u[n] \end{aligned}$$

where

$$\begin{aligned} A_k &= |A_k| e^{j\alpha_k} \\ d_k &= r_k e^{j\beta_k} \end{aligned}$$

- We now have a new transform pair

$$2|A_k| r_k^n \cos(\beta_k n + \alpha_k) u[n] \overset{z}{\longleftrightarrow} \frac{A_k}{1 - d_k z^{-1}} + \frac{A_k^*}{1 - d_k^* z^{-1}},$$

with ROC = $|z| > |d_k| = r_k$

- Observe now that a complex conjugate pole pair with ROC $|z| > |d_k|$ corresponds to a causal sinusoidal sequence with exponential envelope
- The distance of the pole from the origin determines the exponential weighting ($r_k > 1$ growing, $r_k < 1$ decaying)
- The angle the poles make to the positive real axis determines the sinusoidal frequency on the interval $[0, \pi]$

Example 3.11:

Given a causal signal with z-transform

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + \frac{1}{2} z^{-2}}$$

- $X(z)$ is a proper rational function with poles $d_1 = 1/2 \pm j1/2$
- The partial fraction expansion is

$$X(z) = \frac{A_1}{1 - d_1 z^{-1}} + \frac{A_1^*}{1 - d_1^* z^{-1}}$$

where it can be shown that $A_1 = 1/2 - j3/2$

- In polar form we can write

$$
\begin{aligned}
A_1 &= \frac{\sqrt{10}}{2} e^{-j71.565^\circ} \\
d_1 &= \frac{1}{\sqrt{2}} e^{j\pi/4}
\end{aligned}
$$

- Using the z-transform pair defined above we can immediately write

$$
x[n] = \sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi n}{4} - 71.565^\circ\right) u[n]
$$

3.4.3 Power Series expansion

By definition

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\
&= \cdots + x[-2]z^2 + x[-1]z + x[0] \\
&\quad + x[1]z^{-1} + x[2]z^{-2} + \cdots
\end{aligned}
$$

By finding the coefficient of a particular z^{-n} in a series expansion of $X(z)$ we obtain $x[n]$ term-by-term.

- For $X(z) = N(z)$ (i.e. $D(z) = 1$) $x[n]$ can be obtained by inspection

- If $X(z) = N(z)/D(z)$ we may use long division to obtain powers of z^{-1} if the ROC implies a right-sided sequence, and powers of z^1 if the ROC implies a left-sided sequence

Example 3.12:

Given $X(z) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$ find $x[n]$.

- By inspection

$$x[n] = \begin{cases} 1, & n = -2 \\ -1/2, & n = -1 \\ -1, & n = 0 \\ 1/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

- Alternatively,

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Example 3.13:

Inverse transform

$$X(z) = \frac{1}{1 - az^{-1}}, \quad \text{ROC} = |z| > |a|$$

- Since the ROC is the exterior of a circle $\Rightarrow$ $x[n]$ is right-sided, so divide to obtain powers of z^{-1}

$$\begin{array}{r|l} & 1 + \; az^{-1} \; + \; a^2z^{-2} \; + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - \; az^{-1}} \\ & \qquad az^{-1} \\ & \qquad \underline{az^{-1} \; - \; a^2z^{-2}} \\ & \qquad\qquad\quad a^2z^{-2} \end{array}$$

so

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

which implies that

$$\begin{aligned} x[n] &= \delta[n] + a\delta[n-1] + a^2\delta[n-2] + \cdots \\ &= a^n u[n] \end{aligned}$$

Example 3.14:

Find the inverse z-transform of

$$X(z) = \ln(1 + az^{-1}), \;\; \text{ROC} = |z| > |a|$$

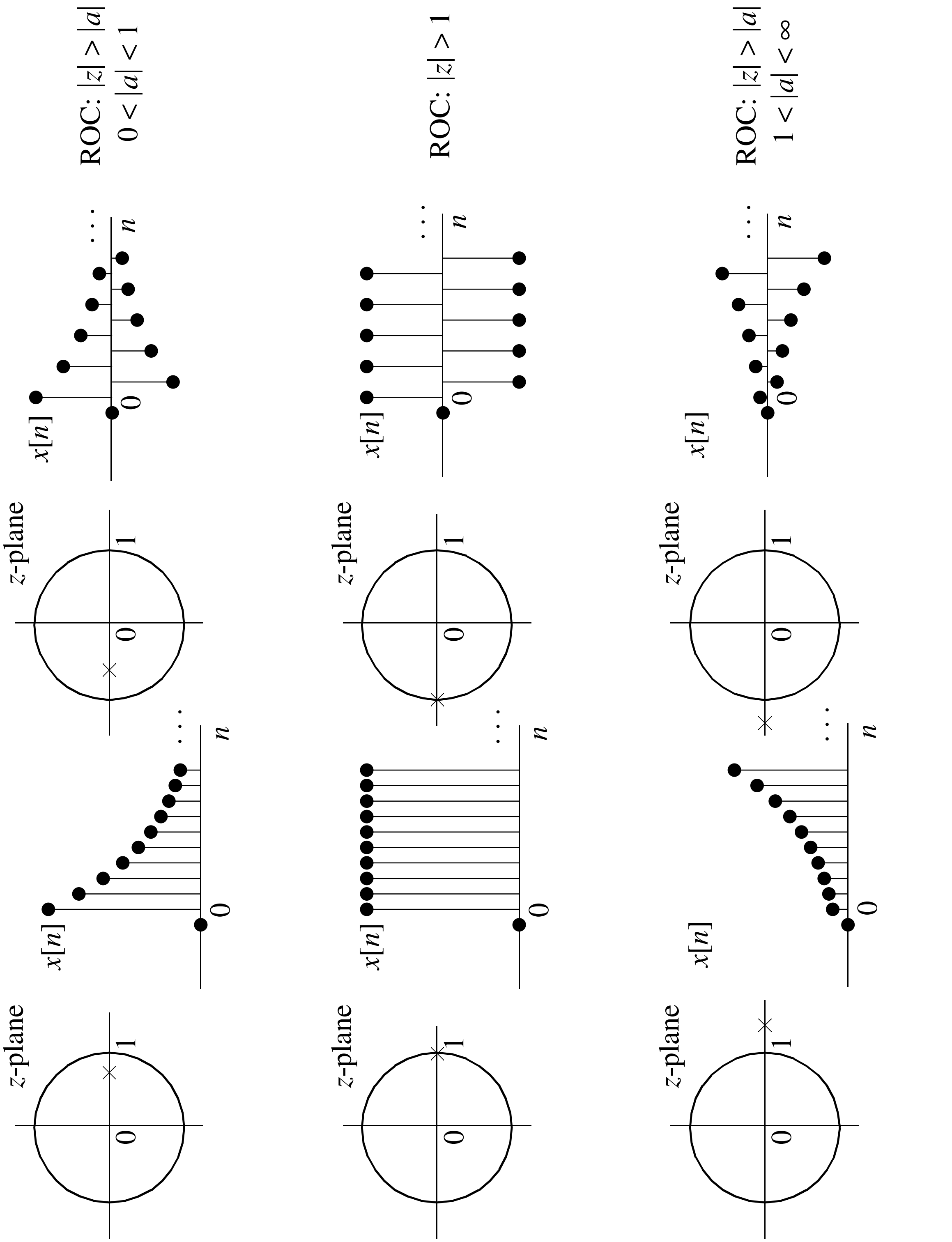
- For $z \in$ ROC we can use the power series expansion for $\ln(1 + x)$ valid for $|x| < 1$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

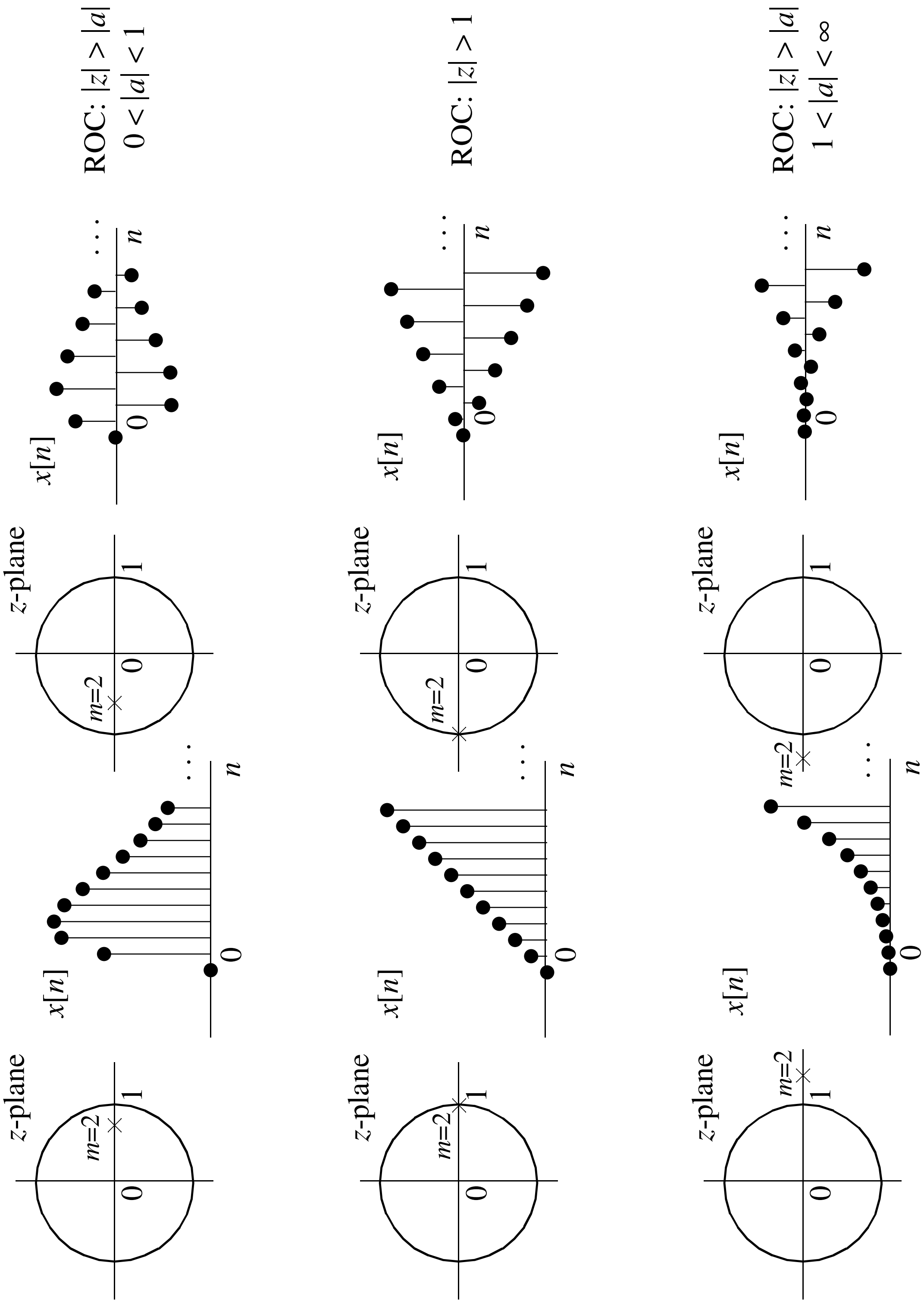
- Inverse transforming term-by-term we obtain

$$x[n] = \begin{cases} (-1)^{n+1}\frac{a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

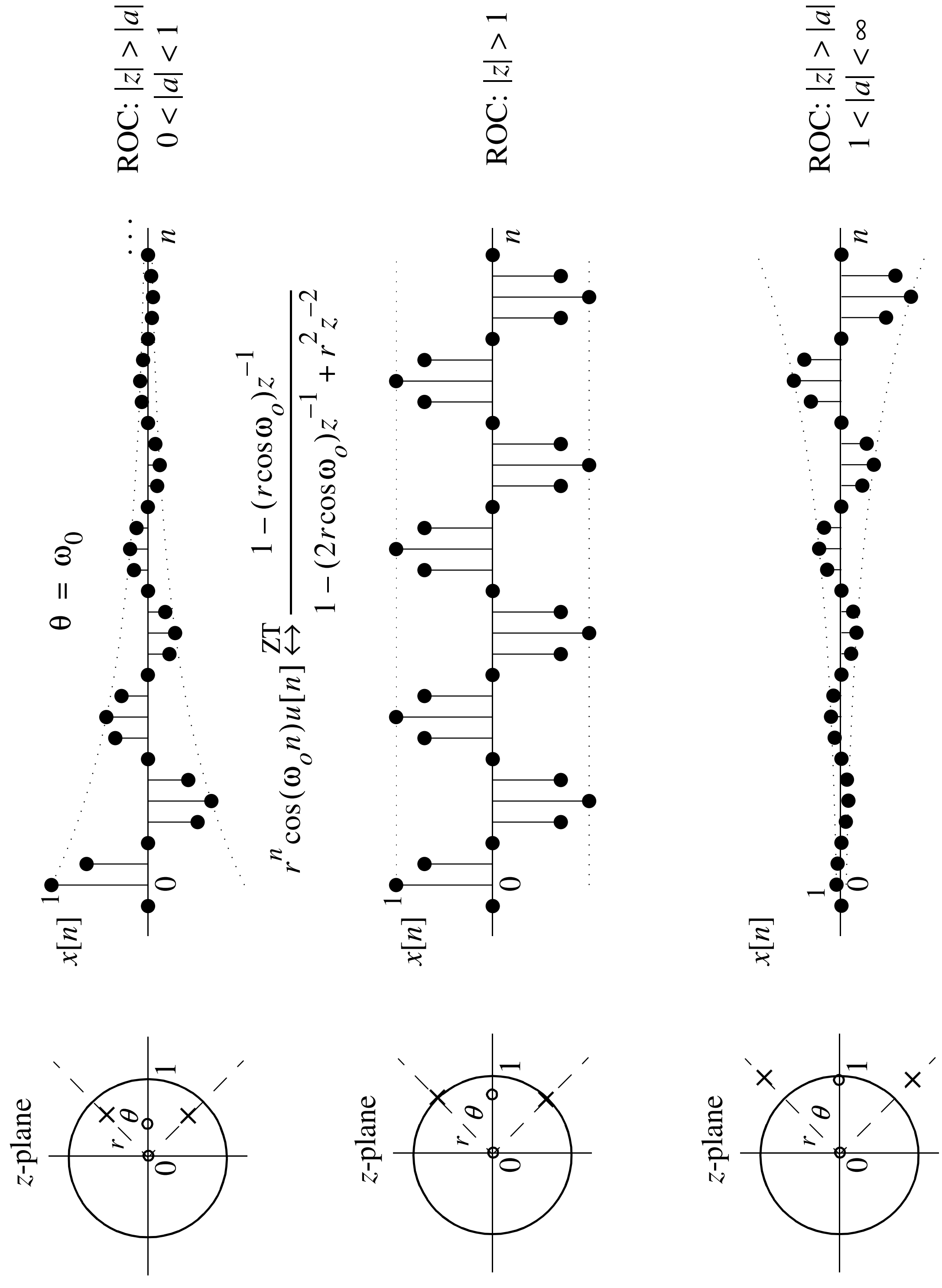
3.4.4 Time-Domain Responses for a Single Real Pole



3.4.5 Time-Domain Responses for a Double Real Pole



3.4.6 Time-Domain Responses for a Conjugate Pole Pair



Example 3.15: The Inverse z-Transform Using Python

- Scipy has supplied functions for performing partial fraction expansions in both the s-domain and the z-domain. The best function for working with z-transforms, `residuez()`, is contained in the signal processing module (`scipy.signal`).

- The help description is the following:

```
Definition: signal.residuez(b, a, tol=0.001, rtype='avg')
Docstring: Compute partial-fraction expansion of b(z) / a(z).

If ``M = len(b)`` and ``N = len(a)``::

          b(z)     b[0] + b[1] z**(-1) + ... + b[M-1] z**(-M+1)
  H(z) = ------ = ----------------------------------------------
          a(z)     a[0] + a[1] z**(-1) + ... + a[N-1] z**(-N+1)

           r[0]                   r[-1]
   = --------------- + ... + ---------------- + k[0] + k[1]z**(-1) ...
     (1-p[0]z**(-1))         (1-p[-1]z**(-1))

If there are any repeated roots (closer than tol), then the partial
fraction expansion has terms like::

        r[i]              r[i+1]                    r[i+n-1]
    -------------- + ------------------ + ... + ------------------
    (1-p[i]z**(-1))  (1-p[i]z**(-1))**2         (1-p[i]z**(-1))**n

See also
--------
invresz, unique_roots
```

- Consider the z-transform

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}},\ \text{ROC:}|z| > 1$$

- Using `residuez()` the solution is

```
In [22]: import scipy.signal as signal
In [23]: R,P,K = signal.residuez([1, 2, 1],[1, -3./2, 1./2])

In [24]: R
Out[24]: array([-9.,  8.])

In [25]: P
Out[25]: array([ 0.5,  1. ])

In [26]: K
Out[26]: array([ 2.])
```

- From the above we can then write that

$$X(z) = 2 + \frac{8}{1 - z^{-1}} + \frac{-9}{1 - 0.5z^{-1}},\ |z| > 1$$

- Inverse transforming term-by-term yields

$$x[n] = 2\delta[n] + 8u[n] - 9(0.5)^n u[n]$$

- As a second example consider the z-transform

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2},\ \text{ROC:}|z| > 1$$

- Using `residuez()` the solution is

```
In [32]: R,P,K = signal.residuez([1],signal.convolve([1,1],
                 signal.convolve([1,-1],[1,-1])))

In [33]: R
Out[33]: array([ 0.25,  0.25,  0.5 ])

In [34]: P
Out[34]: array([-1.,  1.,  1.])

In [35]: K
Out[35]: array([ 0.])
```

- The above results indicate that

$$X(z) = \frac{0.25}{1 - z^{-1}} + \frac{0.5}{(1 - z^{-1})^2} + \frac{0.25}{1 + z^{-1}}, \ |z| > 1$$

- Thus a term-by-term inversion yields

$$x[n] = 0.25u[n] + 0.5(n + 1)u[n] + 0.25(-1)^n u[n]$$

3.5 z-Transform Properties

The z-transform properties considered here parallel those of the Fourier transform. In the following we will assume that

$$x_i[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_i(z), \ \text{ROC} = R_{x_i}, \ i = 1, \ 2, \ldots$$

- Linearity:

$$ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)$$

with ROC $= R_{x_1} \cap R_{x_2}$.

Note: If the sum of $X_1(z)$ and $X_2(z)$ introduces a pole-zero cancellation, then the ROC may be larger than indicated (e.g. rectangular window sequence example notes Example 3.5 or O&S Example 3.6)

- Time-Shifting:

$$x[n-n_o] \stackrel{z}{\longleftrightarrow} z^{-n_o}X(z), \ \ \text{ROC} = R_x$$

Note that $z = 0$ or $|z| \to \infty$ may be added or deleted from the ROC.

The proof follows directly from the definition.

We will see later that for the one-sided z-transform this theorem is modified. This theorem is important in solving LCCDEs with nonzero initial conditions.

Example 3.16:

Find the inverse z-transform of

$$X(z) = \frac{z^{-1}}{1-\frac{1}{4}z^{-1}}, \ \text{ROC} = |z| > \frac{1}{4}$$

- Since $M = N = 1$ to use partial fractions we would first use long division to write

$$X(z) = -4 + \frac{4}{1-\frac{1}{4}z^{-1}}$$

- It immediately follows that

$$\begin{aligned} x[n] &= -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n] \\ &= \underbrace{4\left(\frac{1}{4}\right)^n}_{(\frac{1}{4})^{n-1}} u[n-1] \end{aligned}$$

- Using the shifting theorem we can also approach the problem as

$$x[n] = \mathcal{Z}^{-1}\left\{\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}\right\} = \mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{4}z^{-1}}\right\}_{n\to n-1}$$
$$= \left(\frac{1}{4}\right)^{n-1} u[n-1]$$

- Multiplication by an Exponential:

$$z_0^n x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z/z_0),\ \text{ROC} = |z_0| R_x$$

where the notation used for the new ROC implies that $R_x = \{z : r_R < |z| < r_L\} \Rightarrow \text{ROC} = \{z : |z_0| r_R < |z| < |z_0| r_L\}$.

Multiplication by a positive real z_0^n corresponds to a shrinking or expanding of the z-plane, while multiplication by $e^{j\omega_o}$ corresponds to a rotation in the z-plane by an angle ω_o.

proof

$$\begin{aligned} Y(z) &= \mathcal{Z}\{z_0^n x[n]\} = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x[n](z/z_0)^{-n} = X(z/z_0) \end{aligned}$$

- Differentiation of $X(z)$:

$$nx[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dX(z)}{dz},\ \text{ROC} = R_x$$

Note that $z = 0$ or $|z| \to \infty$ may be added or deleted from the ROC.

proof

$$\begin{aligned} -z\frac{dX(z)}{dz} &= -z \sum_{n=-\infty}^{\infty} (-n)x[n]z^{-n-1} \\ &= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\} \end{aligned}$$

Example 3.17:

Determine the z-transform of the sequence

$$x[n] = na^n u[n] = n(a^n u[n])$$

- Direct application of the theorem yields

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right), \quad |z| > |a|$$

thus

$$na^n u[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{az^{-1}}{(1 - az^{-1})^2}, \quad |z| > |a|$$

- Conjugation of a Sequence:

$$x^*[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X^*(z^*), \quad \text{ROC} = R_x$$

The proof follows from the definition.

- Time Reversal:

$$x[-n] \overset{\mathcal{Z}}{\longleftrightarrow} X(1/z), \quad \text{ROC} = \frac{1}{R_x}$$

where the notation used for the new ROC implies that $R_x = \{z : r_R < |z| < r_L\} \Rightarrow \text{ROC} = \{z : 1/r_L < |z| < 1/r_R\}$.

The proof follows from the definition.

Example 3.18:

Determine the z-transform of $u[-n]$

- We know that

$$u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z^{-1}}, \quad \text{ROC} = |z| > 1$$

- Applying the theorem

$$u[-n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1 - z} = \frac{-z^{-1}}{1 - z^{-1}}, \text{ ROC} = |z| < 1$$

- Convolution of Sequences:

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z)X_2(z)$$

where ROC$=$ $R_{x_1} \cap R_{x_2}$ or larger if pole-zero cancellation occurs *proof*

Let

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Then

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \end{aligned}$$

Now let $n \to m = n - k$ then,

$$\begin{aligned} Y(z) &= \sum_{k=-\infty}^{\infty} x_1[k] \underbrace{\left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\}}_{X_2(z)} z^{-k} \\ &= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} X_2(z) = X_1(z)X_2(z) \end{aligned}$$

Example 3.19:

Let $x_1[n] = a^n u[n]$ and $x_2[n] = u[n]$ and find $y[n] = x_1[n] * x_2[n]$.

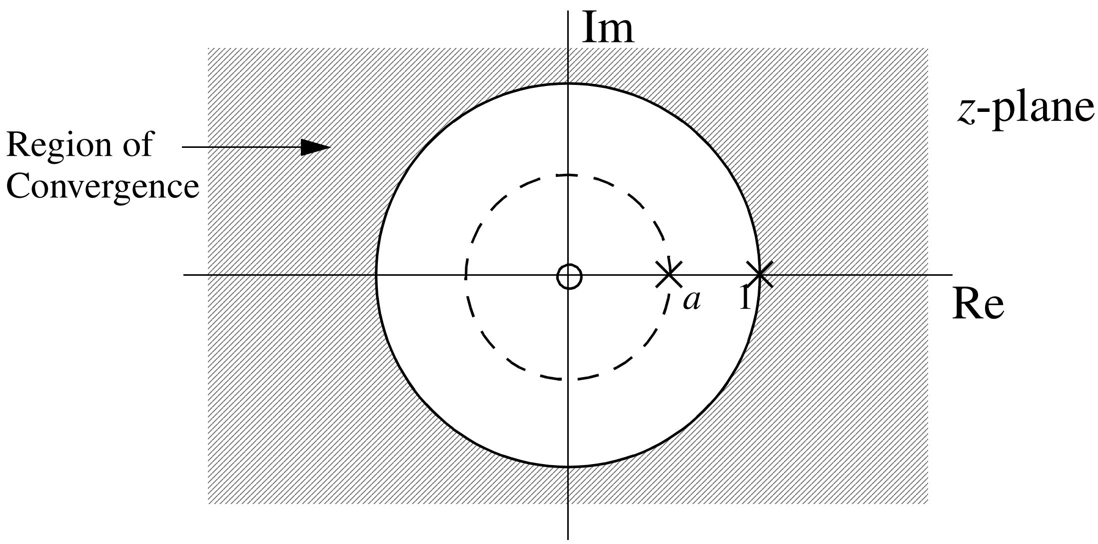
- Using the transform pair tables we can write

$$\begin{aligned} X_1(z) &= \frac{1}{1 - az^{-1}}, \ \text{ROC} = |z| > |a| \\ X_2(z) &= \frac{1}{1 - z^{-1}}, \ \text{ROC} = |z| > 1 \end{aligned}$$

- The transform of $Y(z)$ is

$$Y(z) = \frac{1}{(1-z^{-1})(1-az^{-1})}, \quad \text{ROC} = |z| > \max\{1, |a|\}$$

- Suppose that $|a| < 1$ then the ROC is as shown below



Pole-zero plot and ROC, $|z| > 1$, for $y[n]$

- Using a partial fraction expansion we can write

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \ \text{ROC} = |z| > 1$$

- Term-by-term inversion yields

$$y[n] = \frac{1-a^{n+1}}{1-a}u[n]$$

- Initial Value Theorem: For $x[n]$ a causal sequence (i.e. $x[n] = 0,\ n < 0$), then

$$x[0] = \lim_{z \to \infty} X(z)$$

<u>*proof*</u>

Since $x[n]$ is causal

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Clearly, as $z \to \infty,\ z^{-n} \to 0$ for $n > 0$, thus the result follows

3.6 Inverse z-Transform Using Contour Integration

To develop the contour integration technique for computing the inverse z-transform we will first study some important results from the theory of complex variables. Then the complex inversion integral formula will be developed, followed by a discussion of evaluation techniques using the residue theorem.

3.6.1 Complex Variable Background

Definition: The *neighborhood* of a point z_o in the complex plane is the open circular disk $|z - z_o| < \rho$ where $\rho > 0$ is an arbitrary real constant.

Definition: A function $F(z)$ is *analytic* or regular at a point $z = z_o$ if it is defined at that point and has a derivative at every point in some neighborhood of z_o. Note: If the derivative at z_o exits it must be the same regardless of the direction of approach used in the limit.

Definition: A function having at least one analytic point is called an *analytic function*.

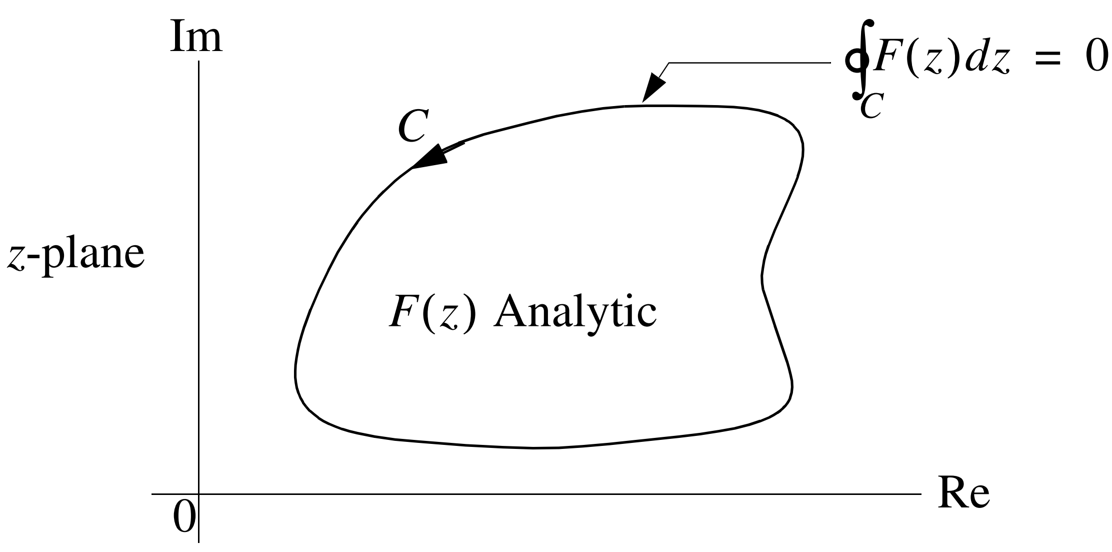
Definition: If the function $F(z)$ is not analytic at z_o, then z_o is called a *singular point*.

3.6.2 Cauchy's Integral Theorem:

Let the function $F(z)$ of complex variable z be analytic everywhere on and within a simple (non-crossing) closed curve C in the z-plane. Then

$$\oint_C F(z)\, dz = 0$$

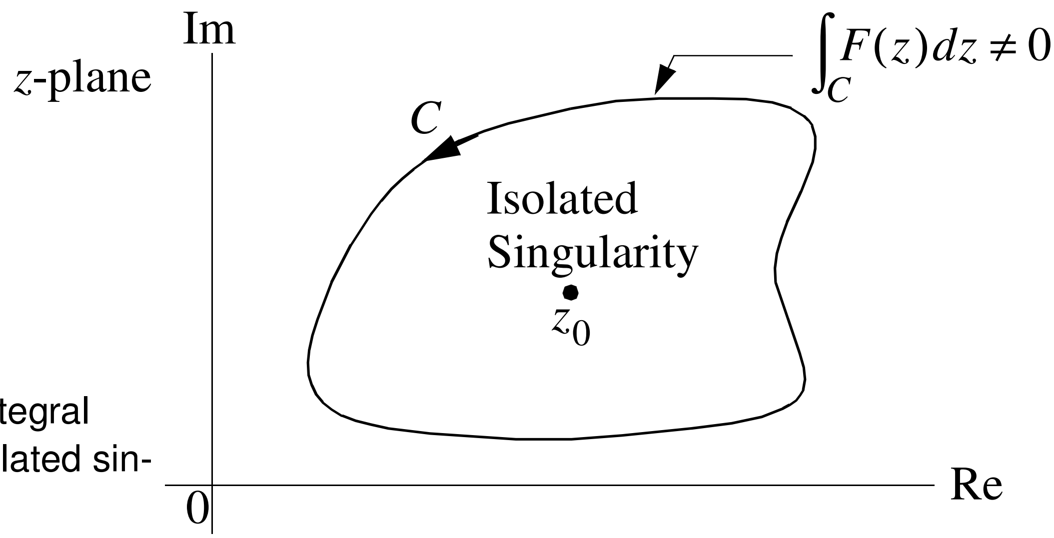
where the line integral is evaluated along <u>any</u> C in a counterclockwise direction.



Cauchy Integral Theorem

- Suppose that $F(z)$ has a singularity at $z = z_o$ which is inside of C, then

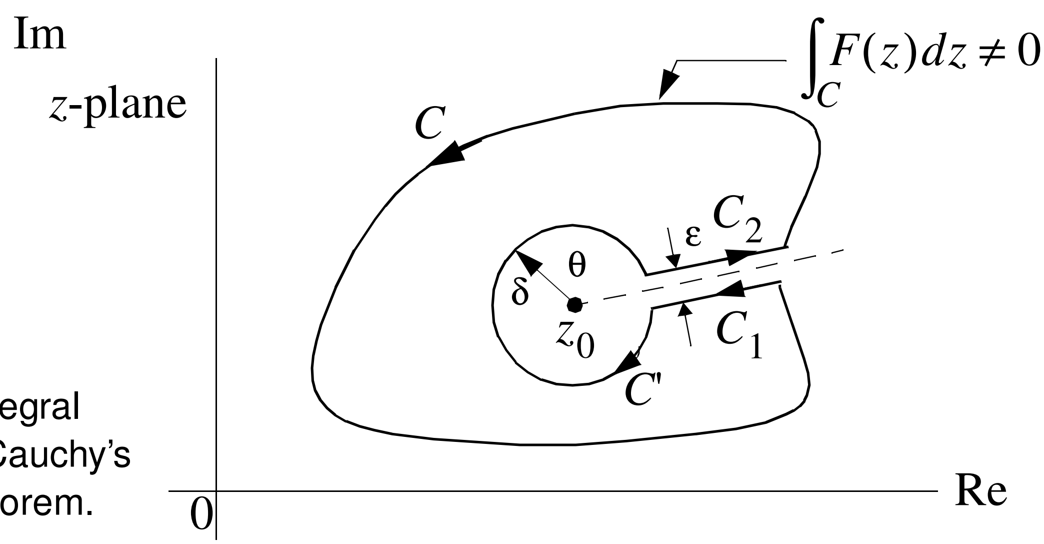
$$\oint_C F(z)\,dz \neq 0$$



Contour integral with an isolated singularity.

- To apply Cauchy's integral theorem we define a new contour $C + C_1 + C' + C_2$ which avoids the singularity at $z = z_o$

$$\oint_C F(z)\,dz + \oint_{C_1} F(z)\,dz + \oint_{C_2} F(z)\,dz + \oint_{C'} F(z)\,dz = 0$$



Contour integral satisfying Cauchy's integral theorem.

- As the gap $\epsilon \to 0$ we see that the contributions from C_1 and C_2 will cancel with the result being

$$\oint_C F(z)\,dz \overset{\epsilon\to 0}{=} -\oint_{C'} F(z)\,dz$$

- Now consider the contour C' which consists of the point set $\{z : z - z_o = \delta e^{j\theta}, 0 \leq \theta < 2\pi\}$

- If we assume that δ is infinitesimal, then we can write

$$F(z) = \frac{G(z)}{z - z_o} \approx \frac{G(z_o)}{z - z_o}, \quad z \in C'$$

  where $G(z)$ is $F(z)$ with the singularity at $z = z_o$ factored out, thus making $G(z)$ analytic at $z = z_o$

- The contour integral over C' as $\delta \to 0$ can now be evaluated using the change of variables $dz \to d(z - z_o) = d(\delta e^{j\theta}) = j\delta e^{j\theta}\,d\theta$,

$$\begin{aligned}\oint_C F(z)\,dz &= \int_0^{2\pi} \frac{G(z_o)}{\delta e^{j\theta}} j\delta e^{j\theta}\,d\theta \\ &= 2\pi j G(z_o)\end{aligned}$$

  The minus sign is removed due to the fact that a clockwise contour has been used (i.e. $[0, 2\pi]$ vs $[2\pi, 0]$)

- If $F(z)$ contains higher order singularities (poles) inside of C, then it can be shown that

$$\begin{aligned}\oint_C F(z)\,dz &= \oint_{C'} \frac{G(z)}{(z - z_o)^s}\,dz \\ &= \frac{2\pi j}{(s-1)!} \underbrace{\left[\frac{d^{s-1}G(z)}{dz^{s-1}}\right]\bigg|_{z=z_o}}_{R_o = \text{ residue at } z = z_o}\end{aligned}$$

which is really just a more general form of Cauchy's integral theorem

3.6.3 Cauchy's Residue Theorem:

Let $F(z)$ be analytic everywhere on and within a simple closed contour C except at isolated singular points $z_1,\ z_2, \ldots, z_N$. Then

$$\oint_C F(z)\,dz = 2\pi j \sum_{n=1}^{N} R_n$$

where the R_n's are the residues of $F(z) = G(z)/(z-z_n)^s$ at $z = z_n$, $n = 1, \ldots, N$.

3.6.4 Complex Inversion Integral

By definition

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- To obtain the inverse z-transform multiply both sides of the above equation by $z^{k-1}/(2\pi j)$ and integrate counterclockwise around a contour
  which encloses the origin and also lies within the ROC of $X(z)$

$$\frac{1}{2\pi j}\oint_C X(z)z^{k-1}\,dz = \frac{1}{2\pi j}\oint_C \sum_{n=-\infty}^{\infty} x[n]z^{-n+k-1}\,dz$$

- The above equation may be reduced on the right side using a special case of the Cauchy integral theorem (i.e. $F(z) =$

$z^{-k}/(2\pi j))$

$$\frac{1}{2\pi j}\oint_C z^{-n+k-1}\,dz = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

<u>*proof:*</u>

- To begin with we assume that the contour C encircles the origin in a counter clockwise fashion. Now from Cauchy's residue theorem we know that

$$\frac{1}{2\pi j}\oint_C F(z)\,dz = \sum \text{Residues of } F(z) \text{ inside } C$$

Here $F(c) = z^{-k}$

- For $k = 1$, $F(z) = G(z)/(z - 0) = 1/z$, which implies that $F(z)$ has a single first-order residue at $z = 0$, thus

$$\frac{1}{2\pi j}\oint_C F(z)\,dz = G(0) = 1$$

- For $k > 1$, $F(z) = G(z)/z^k = 1/z^k$, which implies that $F(z)$ has a single kth-order residue at $z = 0$, thus

$$\frac{1}{2\pi j}\oint_C F(z)\,dz = \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dz^{k-1}}1\right]\Bigg|_{z=0} = 0$$

- Finally we conclude that

$$\frac{1}{2\pi j}\oint_C z^{-k}\,dz = \begin{cases} 1, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

- The first step in reducing the right side is to interchange the order of integration and summation (valid since the series converges on the chosen contour)

$$\begin{aligned} \frac{1}{2\pi j}\oint_C X(z)z^{k-1}\,dz &= \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi j}\oint_C z^{-n+k-1}\,dz \\ &= x[k] \end{aligned}$$

- Turning the above relation around we have the inverse z-transform

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$$

- Note in particular if the ROC includes the unit circle, then letting $C \to z = e^{j\omega}$, results in $dz \to je^{j\omega}\,d\omega$, and

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

  which is the inverse Fourier transform expression

3.6.5 Complex Inversion Integral Evaluation using the Residue Theorem

From the Cauchy residue theorem we can write that

$$x[n] = \sum[\text{residues of } X(z)z^{n-1} \text{ at the poles inside } C]$$

- If $X(z)$ is a rational function in z then we may write $X(z)z^{n-1}$ as

$$X(z)z^{n-1} = \frac{\psi(z)}{(z-d_o)^s}$$

  for a pole of order s in the product $X(z)z^{n-1}$ and $\psi(z)$ analytic at $z = d_o$

- The general formula (from Cauchy's integral theorem) is

$$\text{Res}[X(z)z^{n-1} \text{ at } z = d_o] = \frac{1}{(s-1)!}\left[\frac{d^{s-1}\psi(z)}{dz^{s-1}}\right]_{z=d_o}$$

- For first-order poles the simpler result

$$\text{Res}[X(z)z^{n-1} \text{ at } z = d_o] = \psi(d_o)$$

  may be used

- Note that finding residues of $X(z)z^{n-1}$ is basically the same as finding the partial fraction expansion coefficients

- A common complication that arises when using the residue method to solve for $x[n]$ is the presence of multiple order poles at $z = 0$ for $n < 0$ (e.g. $X(z)z^{n-1} = \psi(z)z^n/(z - d_o)$)

- A solution is to perform a transformation in the complex inversion integral by letting $z \to 1/p$, then we have

$$\begin{aligned} x[n] &= -\frac{1}{2\pi j}\oint_{C'} X(1/p)(1/p)^{n-1}\left(-\frac{1}{p^2}\right) dp \\ &= \frac{1}{2\pi j}\oint_{C'} X(1/p)p^{-n-1}\, dp \\ &= \sum \text{Res}[X(1/p)p^{-n-1} \text{ at poles inside } C'] \end{aligned}$$

  where C' is a circle of radius $1/r$ in the p-plane if C is a circle of radius r in the z-plane and poles outside of C in the z-plane are now poles inside of C' in the p-plane

- Note the extra minus sign is used to make a clockwise contour (from $z \to 1/p$) into a more familiar counterclockwise contour

Example 3.20:

Using the complex inversion integral find the inverse z-transform of

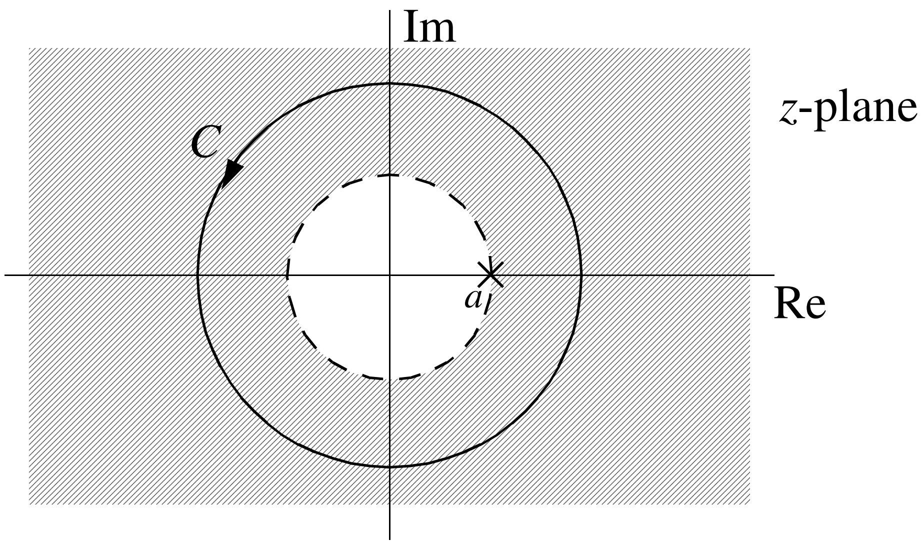
$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

- In terms of the complex inversion integral $x[n]$ is

$$x[n] = \frac{1}{2\pi j} \oint_C \frac{z^{n-1}}{1 - az^{-1}}\, dz = \frac{1}{2\pi j} \oint_C \frac{z^n\, dz}{z - a}$$

  where C is chosen to be a circle with radius greater than $|a|$

- For $n \geq 0$ the contour encloses a simple pole at $z = a$

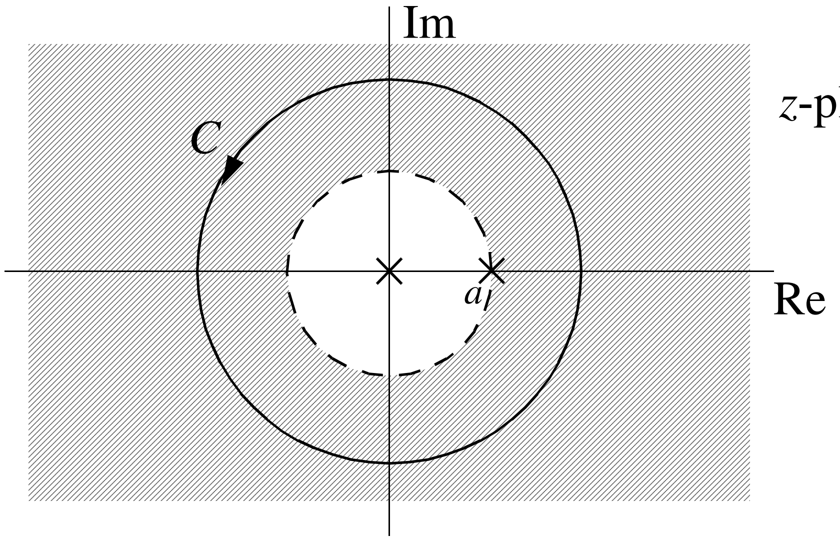


z-plane showing C and the poles of $X(z)z^{n-1}$ for $n \geq 0$

- The residue of the pole at $z = a$ is simply a^n, thus

$$x[n] = a^n, \quad n \geq 0$$

- For $n < 0$ the contour encloses a simple pole at $z = a$ and a multiple pole (of order $s = -n$) at $z = 0$, thus two residues must be evaluated



z-plane showing C and the poles of $X(z)z^{n-1}$ for $n < 0$

- To avoid the tedium of evaluating the multiple-order pole at $z = 0$, we can use the variable change $z \to 1/p$ in the contour integral. Thus,

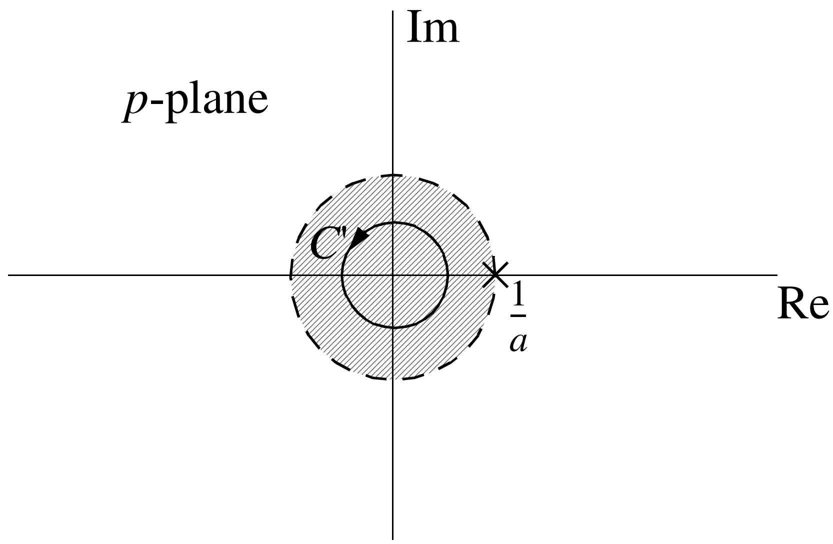
$$X(1/p) = \frac{1}{1 - ap}, \quad |p| < |1/a|$$

and

$$x[n] = \frac{1}{2\pi j} \oint_{C'} \frac{p^{-n-1}}{1 - ap}\, dp$$

- The new contour, C', is a circle of radius less than $|1/a|$
- For $n < 0$ there are no singularities within C', thus

$$x[n] = 0, \quad n < 0$$



p-plane showing C' and the poles of $X(1/p)p^{-n-1}$ for $n < 0$

- In summary

$$x[n] = a^n u[n]$$

Example 3.21:

Find the inverse z-transform of

$$X(z) = \frac{z^3 + 2z^2 + 3z}{(z - \frac{1}{2})(z^2 - 2z + 4)}, \quad |z| > 2$$

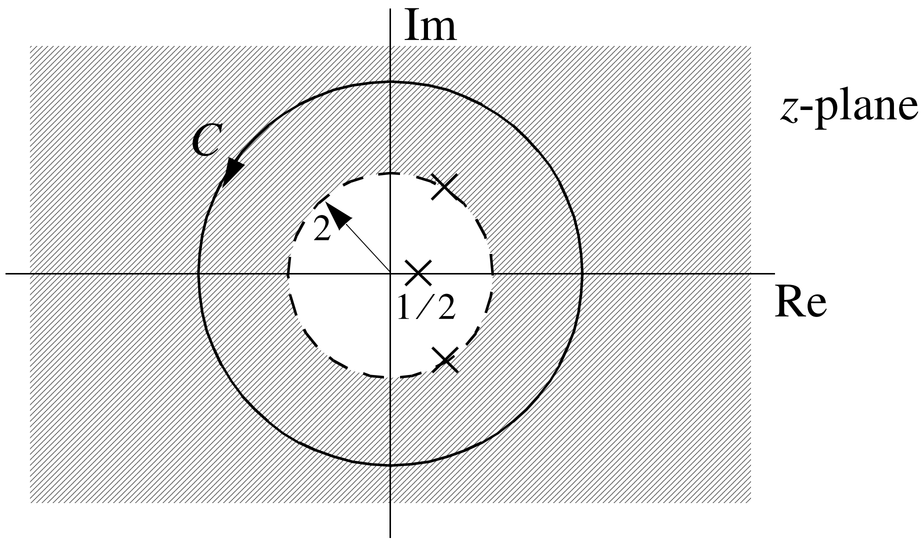
using contour integration.

- The contour integral we must evaluate is

$$x[n] = \frac{1}{2\pi j} \oint_C \frac{z^n(z^2 + 2z + 3)}{(z - \frac{1}{2})(z^2 - 2z + 4)} \, dz$$

- Let C be a circle with radius greater than 2

- For $n \geq 0$ we have three poles within C: $z_1 = 1/2$, $z_2 = 1 + j\sqrt{3}$, and $z_3 = 1 - j\sqrt{3}$



z-plane showing C and the poles of $X(z)z^{n-1}$ for $n \geq 0$

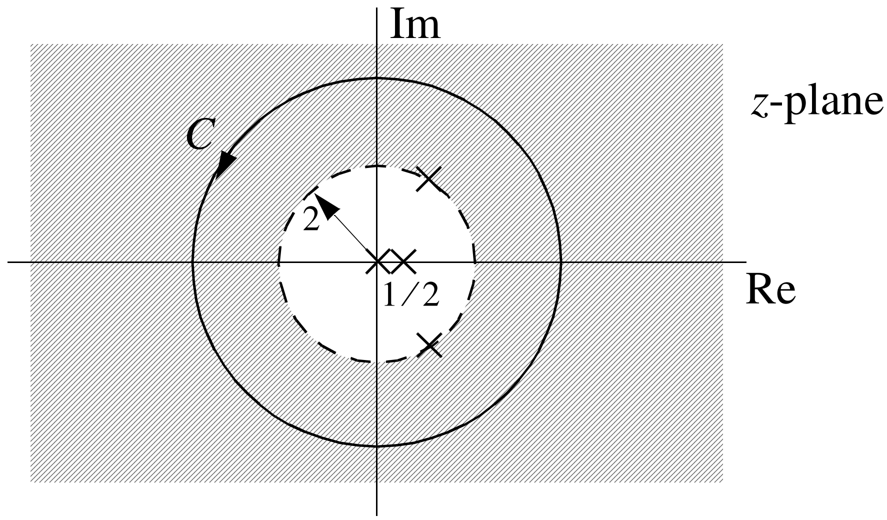
- The corresponding residues are:

$$\begin{aligned} R_1 &= z^n \left.\frac{z^2 + 2z + 3}{z^2 - 2z + 4}\right|_{z=\frac{1}{2}} \\ &= \frac{17}{13}\left(\frac{1}{2}\right)^n \\ R_2 &= z^n \left.\frac{z^2 + 2z + 3}{(z - \frac{1}{2})[z - (1 - j\sqrt{3})]}\right|_{z=1+j\sqrt{3}} \\ &= \frac{-1 - j9\sqrt{3}}{13}(1 + j\sqrt{3})^n \\ R_3 &= R_2^* = \frac{-1 + j9\sqrt{3}}{13}(1 - j\sqrt{3})^n \end{aligned}$$

thus

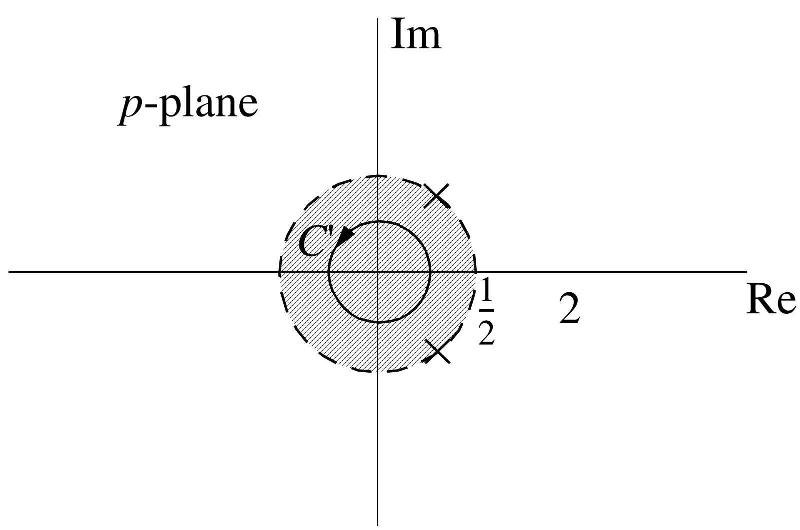
$$x[n] = R_1 + R_2 + R_3, \quad n \geq 0$$

- For $n < 0$ we have a multiple-order pole at $z = 0$ in addition to the same three poles present for $n \geq 0$



z-plane showing C and the poles of $X(z)z^{n-1}$ for $n < 0$

- If we change variables in the contour integral using $z \to 1/p$, then the new contour, C', becomes a circle with radius less than $1/2$ in the p-plane and the poles are now at $p_1 = 2$, $p_2 = (1 + j\sqrt{3})^{-1}$, and $p_3 = (1 - j\sqrt{3})^{-1}$



p-plane showing C and the poles of $X(1/p)p^{-n-1}$ for $n < 0$

- Note that for $n < 0$ there are no poles within C', thus

$$x[n] = 0, \quad n < 0$$

- Summarizing we can write

$$\begin{aligned} x[n] &= \left[\frac{17}{13}\left(\frac{1}{2}\right)^n + \left(\frac{-2 - j9\sqrt{3}}{13}\right)(1 + j\sqrt{3})^n \right. \\ &\quad \left. + \left(\frac{-2 + j9\sqrt{3}}{13}\right)(1 - j\sqrt{3})^n\right] u[n] \end{aligned}$$

- Note that the complex exponentials can be simplified by first writing

$$\frac{-2 - j9\sqrt{3}}{13} = 1.209e^{-97.3^\circ} \text{ and } 1 + j\sqrt{3} = 2e^{j\pi/3}$$

- Using the results for conjugate pole pairs found on page 3-27 of the notes we can write

$$x[n] = \left\{\frac{17}{13}\left(\frac{1}{2}\right)^n + 2.418(2)^n \cos\left[\frac{\pi}{3}n - 97.3^\circ\right]\right\} u[n]$$

3.7 Contour Integration Based Thrms.

In this section special z-transform theorems that rely on contour integration are presented. For most applications these theorems find little use, but there are occasions where using them can be very helpful.

3.7.1 Complex Convolution Theorem

The modulation or multiplication theorem as developed in Chapter 2 page 2-70 resulted in a periodic convolution in the Fourier transform domain. In the z-transform this result generalizes to the complex convolution theorem.

The theorem states that given

$$x_1[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \text{ and } x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_2(z)$$

then

$$\begin{aligned} W(z) &= \mathcal{Z}\{z_1[n]z_2[n]\} \\ &= \frac{1}{2\pi j}\oint_{C_2} X_1(z/v)X_2(v)v^{-1}\,dv \end{aligned}$$

where the contour C_2 is chosen to lie in the intersection of the ROCs of $X_1(z/v)$ and $X_2(v)$.

- To find the ROC for $W(z)$ first consider the proper region for C_2. Suppose that $x_1[n]$ and $x_2[n]$ have the following ROCs

$$\text{ROC}_{x_1} : r_{R1} < |z| < r_{L1}$$

$$\text{ROC}_{x_2} : r_{R2} < |z| < r_{L2}$$

- The contour C_2 must lie in the v-plane such that

$$r_{R1} < \left|\frac{z}{v}\right| < r_{L1} \text{ and } r_{R2} < |v| < r_{L2}$$

thus since $|v| r_{R1} < |z| < |v| r_{L1}$, the ROC of $W(z)$ must at least be

$$\text{ROC}_w : r_{R1} r_{R2} < |z| < r_{L1} r_{L2}$$

It may be a larger region if pole-zero cancellations occur.

proof

$$\begin{aligned} W(z) &= \sum_{n=-\infty}^{\infty} x_1[n] x_2[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x_1[n] \left[\frac{1}{2\pi j} \oint_{C_2} X_2(v) v^{n-1}\, dv\right] z^{-n} \\ &= \frac{1}{2\pi j} \oint_{C_2} \left[\sum_{n=-\infty}^{\infty} x_1[n] (z/v)^{-n}\right] X_2(v) v^{-1}\, dv \\ &= \frac{1}{2\pi j} \oint_{C_2} X_1(z/v) X_2(v) v^{-1}\, dv \end{aligned}$$

- If C_2 can be taken on the unit circle, and $W(z)$ can also be evaluated on the unit circle, then the resulting $W(e^{j\omega})$ is the periodic convolution of Fourier transforms as seen in Chapter 2

Example 3.22:

Let $x_1[n] = a^n u[n]$ and $x_2[n] = b^n u[n]$ and find the z-transform of $w[n] = x_1[n]x_2[n]$ using the complex convolution theorem.

- To begin with we need $X_1(z)$ and $X_2(z)$

$$X_1(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$X_2(z) = \frac{1}{1 - bz^{-1}}, \quad |z| > |b|$$

- Substitution into the contour integral gives

$$\begin{aligned} W(z) &= \frac{1}{2\pi j} \oint_{C_2} \frac{z/v}{(z/v - a)} \frac{v}{v - b} v^{-1}\, dv \\ &= \frac{1}{2\pi j} \oint_{C_2} \frac{-(z/a)}{(v - z/a)} \frac{1}{v - b}\, dv \end{aligned}$$

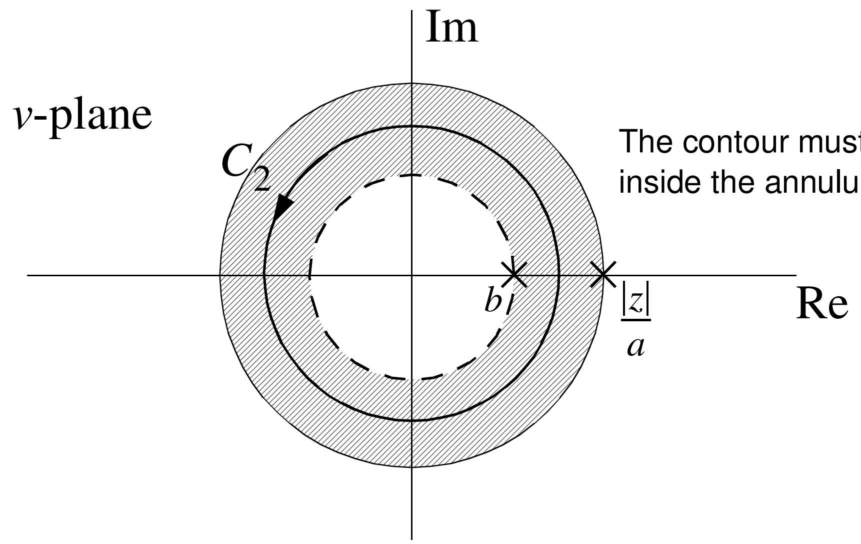
- Note that we have poles at $v = b$ and $v = z/a$

- The contour C_2 must be chosen such that

$$|v| > |b| \text{ and } \left|\frac{z}{v}\right| > |a|$$

or

$$|b| < |v| < \left|\frac{z}{a}\right|$$



The relation between the poles of the integrand and C_2

- The ROC of $W(z)$ is simply $|z| > |ab|$
- With C_2 as shown above we see that only the pole at $v = b$ contributes a residue, thus

$$W(z) = \left.\frac{-z/a}{v - z/a}\right|_{v=b} = \frac{1}{1 - abz^{-1}},\ |z| > |ab|$$

3.7.2 Parseval's Relation

Parseval's relation generalized to the z-transform states that for complex sequences $x_1[n]$ and $x_2[n]$

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}\,dv$$

where the contour C is chosen to lie in the intersection of the ROC of $X_1(v)$ and $X_2^*(1/v^*)$.

Special Case: As a special case suppose $x_1[n] = x_2[n] = x[n]$, and $x[n]$ is also real. Under these assumptions Parseval's relation becomes

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi j} \oint_C X(v)X(v^{-1})v^{-1}\,dv$$

where C must lie in the ROC of $X(z)$.

- *proof*

  To begin with define the sequence $y[n]$ as

  $$y[n] = x_1[n]x_2^*[n]$$

  The z-transform of $y[n]$ is

  $$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]z^{-n} \\ &= \frac{1}{2\pi j} \oint_C X_1(v)X_2^*(z^*/v^*)v^{-1}\,dv \end{aligned}$$

  with the last line a direct result of the complex convolution theorem.

  If we let $z = 1$ in both quantities on the right (assuming the ROC of $Y(z)$ contains the unit circle) we obtain

  $$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j} \oint_C X_1(v)X_2^*(1/v^*)v^{-1}\,dv$$

  which is the desired result.

Example 3.23:

Suppose we have a right-sided real sequence $x[n]$ with z-transform

$$X(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})}, \quad |a|, \ |b| < 1$$

Find the energy in the sequence $x[n]$.

- From the special case of Parseval's relation we can write

$$\begin{aligned} E_x &= \sum_{n=-\infty}^{\infty} x^2[n] \\ &\stackrel{\text{also}}{=} \frac{1}{2\pi j} \oint_C \frac{dv}{(1 - av^{-1})(1 - bv^{-1})(1 - av)(1 - bv)v} \\ &= \frac{1}{2\pi j} \oint_C \frac{v\,dv}{(v - a)(v - b)(1 - av)(1 - bv)} \end{aligned}$$

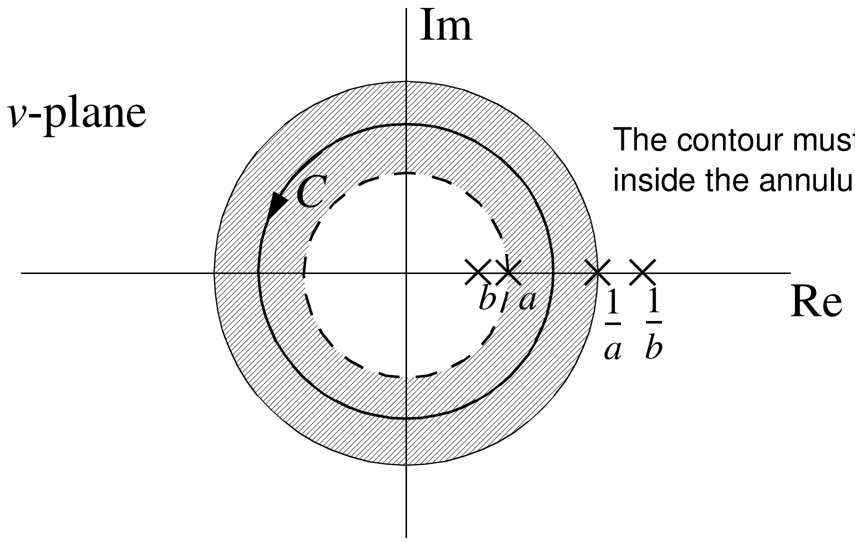
- Note that there are poles at $v = a, \ b, \ a^{-1}$, and b^{-1}

- The contour C must satisfy the conditions

$$|v| > \max\{a, b\} \ \text{ and } \ |v^{-1}| > \max\{a, b\}$$

or

$$\max\{a, b\} < |v| < \min\{1/a, 1/b\}$$



The relation between the poles of the integrand and C_2

- Only the poles at $v = a$ and $v = b$ are inside the contour, thus from the residue theorem

$$
\begin{aligned}
E_x &= \left.\frac{v}{(v-b)(1-av)(1-bv)}\right|_{v=a} \\
&\quad + \left.\frac{v}{(v-a)(1-av)(1-bv)}\right|_{v=b} \\
&= \frac{a(1-b^2)-b(1-a^2)}{(a-b)(1-a^2)(1-b^2)(1-ab)}
\end{aligned}
$$

3.8 Unilateral z-Transform

Recall that the unilateral z-transform was defined as

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- If $x[n] = 0$ for $n < 0$, then the bilateral and unilateral z-transforms are equivalent

- The ROC of the unilateral transform is of the same general form as the ROC of a bilateral transformed right-sided sequence, i.e.

$$\text{ROC} = |z| > r_R = \text{maximum pole radius}$$

Example 3.24:

Find both the bilateral and unilateral z-transforms of $x[n] = \delta[n - n_o]$.

- For $n_o \geq 0$

$$X(z) = z^{-n_o} \text{ and } \mathcal{X}(z) = z^{-n_o}$$

- For $n_o < 0$

$$X(z) = z^{-n_o} \text{ and } \mathcal{X}(z) = 0$$

One of the primary uses of the unilateral z-transform is in solving LCCDEs with non-zero initial conditions. Realizable LTI systems are generally implemented using an LCCDE representation with causality assumed. The system output we desire is thus of the form $y[n]$ $n \geq 0$ for prescribed initial conditions.

The linearity and time shift properties are both needed in the solution of LCCDEs using the unilateral z-transform.

3.8.1 Time-Shifting Property

The time-shifting property changes when going from the bilateral to the unilateral transform. To investigate this property let $y[n] = x[n-n_o]$, then assuming $x[n] \leftrightarrow \mathcal{X}(z)$, find $\mathcal{Y}(z)$.

- case 1: $n_o \geq 0$ (time delay)

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n-n_o]z^{-n} \\
&= z^{-n_o} \sum_{k=-n_o}^{-1} x[k]z^{-k} + z^{-n_o} \sum_{k=0}^{\infty} x[k]z^{-k} \\
&= \sum_{k=1}^{n_o} x[k-1-n_o]z^{-k+1} + z^{-n_o}\mathcal{X}(z)
\end{aligned}$$

- case 2: $n_o < 0$ (time advance)

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n-n_o]z^{-n} = z^{-n_o} \sum_{k=-n_o}^{\infty} x[k]z^{-k} \\
&= z^{-n_o} \sum_{k=0}^{\infty} x[k]z^{-k} - z^{-n_o} \sum_{k=0}^{-n_o-1} x[k]z^{-k} \\
&= z^{-n_o}\mathcal{X}(z) - z^{-n_o} \sum_{k=0}^{-n_o-1} x[k]z^{-k}
\end{aligned}$$

For $n_o \geq 0$ the shifting property can be viewed as follows

$$\begin{aligned}\mathcal{Y}(z) &= x[-n_o] + x[-n_o+1]z^{-1} + \cdots + x[-1]z^{-n_o+1} \\ &\quad + z^{-n_o}\mathcal{X}(z)\end{aligned}$$

We see that by shifting $x[n]$ by n_o samples to the right, n_o *new* samples enter the positive time axis. The transform of these terms corresponds to the first n_o terms in $\mathcal{Y}(z)$ as given above. The *old* samples of $x[n]$ shifted to the right by n_o have transform $z^{-n_o}\mathcal{X}(z)$ as expected from the bilateral time-shift property.

3.8.2 Solution of Causal LCCDEs

An Nth order causal system with LCCDE representation can be written as

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r]$$

with initial conditions $y[-N],\ y[-N+1], \ldots, y[-1]$ and $x[-M]$, $x[-M+1], \ldots, x[-1]$. We can solve this equation using the unilateral z-transform.

- To begin with take the unilateral z-transform of both sides of the above equation using the linearity property and the time-shift property

$$\mathcal{Y}(z)\sum_{k=0}^{N} a_k z^{-k} + \underbrace{\sum_{k=1}^{N} a_k z^{-k} \sum_{m=-k}^{-1} y[m] z^{-m}}_{\mathcal{Y}_i(z)}$$

$$= \mathcal{X}(z)\sum_{r=0}^{M} b_r z^{-r} + \underbrace{\sum_{r=1}^{M} b_r z^{-r} \sum_{m=-r}^{-1} x[m]z^{-m}}_{\mathcal{X}_i(z)}$$

- Solve for $\mathcal{Y}(z)$:

$$\mathcal{Y}(z) = \mathcal{X}(z)\frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}} - \frac{\mathcal{Y}_i(z)}{\sum_{k=0}^{N} a_k z^{-k}} + \frac{\mathcal{X}_i(z)}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Finally to obtain $y[n]$ inverse z-transform $\mathcal{Y}(z)$

Example 3.25:

Find the step response of the first order system

$$y[n] = ay[n-1] + x[n], \quad |a| < 1$$

with initial condition $y[-1] = c$

- Taking the unilateral z-transform of both sides we obtain

$$\mathcal{Y}(z) = a\{z^{-1}\mathcal{Y}(z) + y[-1]\} + \mathcal{X}(z)$$

- Solving for $\mathcal{Y}(z)$ using $y[-1] = c$ and $\mathcal{X}(z) = 1/(1 - z^{-1})$ results in

$$\mathcal{Y}(z) = \frac{ac}{1 - az^{-1}} + \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

- Inverse transforming using partial fractions gives

$$y[n] = ca^{n+1}u[n] + \frac{1-a^{n+1}}{1-a}u[n]$$

- The Scipy `signal.lfilter()` function can also be used to evaluate LCCDEs with nonzero initial conditions

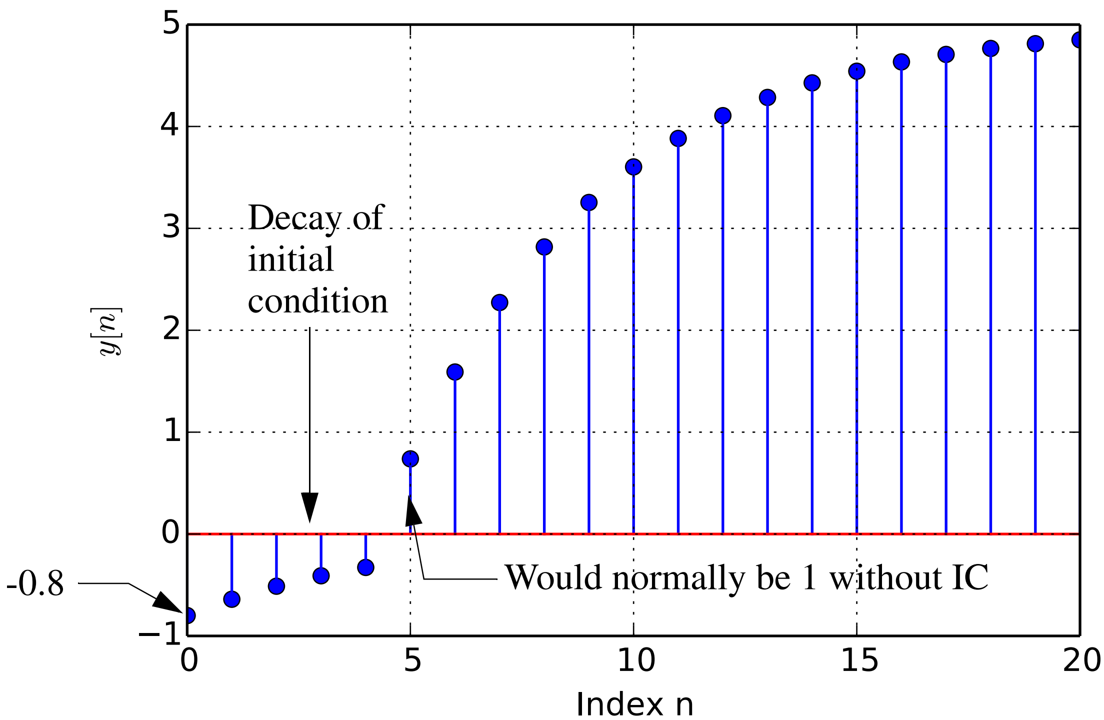
- The filter topology used by the `lfilter()` is what is known as *Transposed Direct Form II*, which will be discussed in Chapter 6

  – The initial state stored in transposed Direct Form II delay elements does not correspond to the standard form of this section

  – For the case of a first-order system we need to store $ay[n-1]$ as opposed to just $y[n-1]$

- To see how this works we will demonstrate with $a = 0.8$, $y[n-1] = -1$, and $x[n] = u[n-5]$

```
In [53]: n = arange(0,21)
In [54]: x = ssd.dstep(n-5)
In [55]: y,zf = signal.lfilter([1],[1,-0.8],x,zi=[-1*0.8])
In [56]: figure(figsize=(6,4))
In [57]: stem(n,y)
In [58]: grid()
In [59]: xlabel('Index n')
In [60]: ylabel(r'$y[n]$')
In [61]: savefig('ch3note_fig29_export.pdf')
```



The output $y[n]$ with $a = 0.8$ and $y[n-1] = -1$

.