Chapter 5

Transform Analysis of Linear Time-Invariant Systems

Contents

Up to this point we have concentrated on the development of the Fourier transform and z-transform as tools for the analysis of discrete-time systems. In this chapter a more thorough investigation of LTI system properties using both Fourier and z-transforms is undertaken.

- In the time domain we have seen that an LTI system is completely characterized by its *impulse response*, $h[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The Fourier transform of $h[n]$, if it exists, gives us the *frequency response* as a complete characterization of an LTI system

$$X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega})$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- The z-transform of $h[n]$, within the appropriate region of convergence, gives us the *system function* as a complete characterization of an LTI system

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

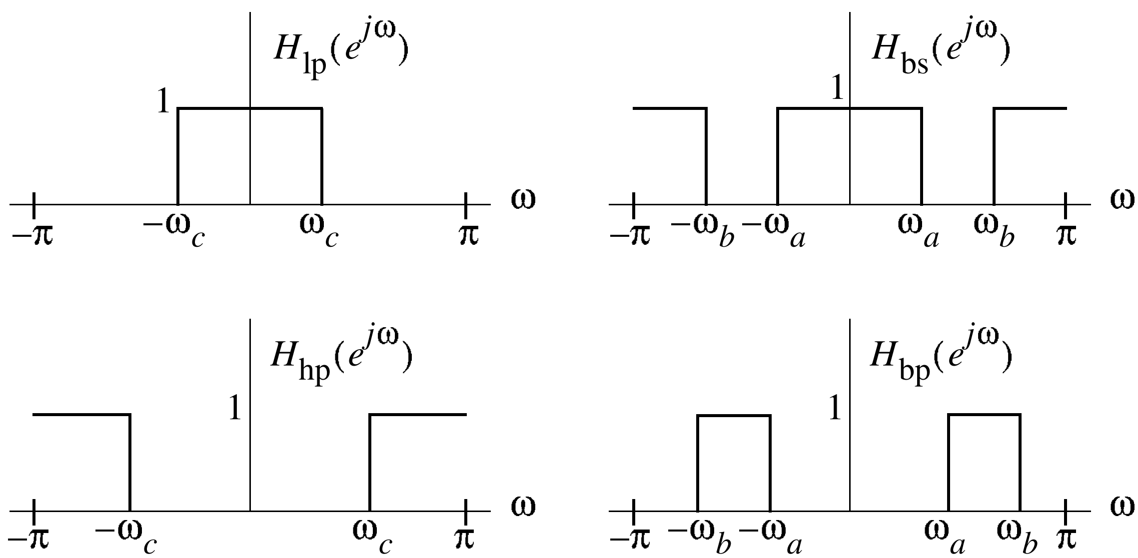
$$Y(z) = H(z)X(z)$$

5.1 Frequency Response of LTI Systems

We have previously defined $H(e^{j\omega})$ in polar form to consist of the magnitude response, $|H(e^{j\omega})|$, and the phase response, $\angle H(e^{j\omega})$.

- The effect $|H(e^{j\omega})|$ has on a signal is often referred to as *magnitude(amplitude) distortion*
- The effect $\angle H(e^{j\omega})$ has on a signal is often referred to as *phase distortion*

5.1.1 Ideal Frequency Selective Filters



Ideal filters over the fundamental period. (a) lowpass. (b) highpass. (c) bandstop. (d) bandpass.

- Note that the phase response is taken to be zero
- Ideal filters have impulse response extending from $-\infty$ to $+\infty$, thus making them computationally unrealizable

5.1.2 Paley-Wiener Theorem

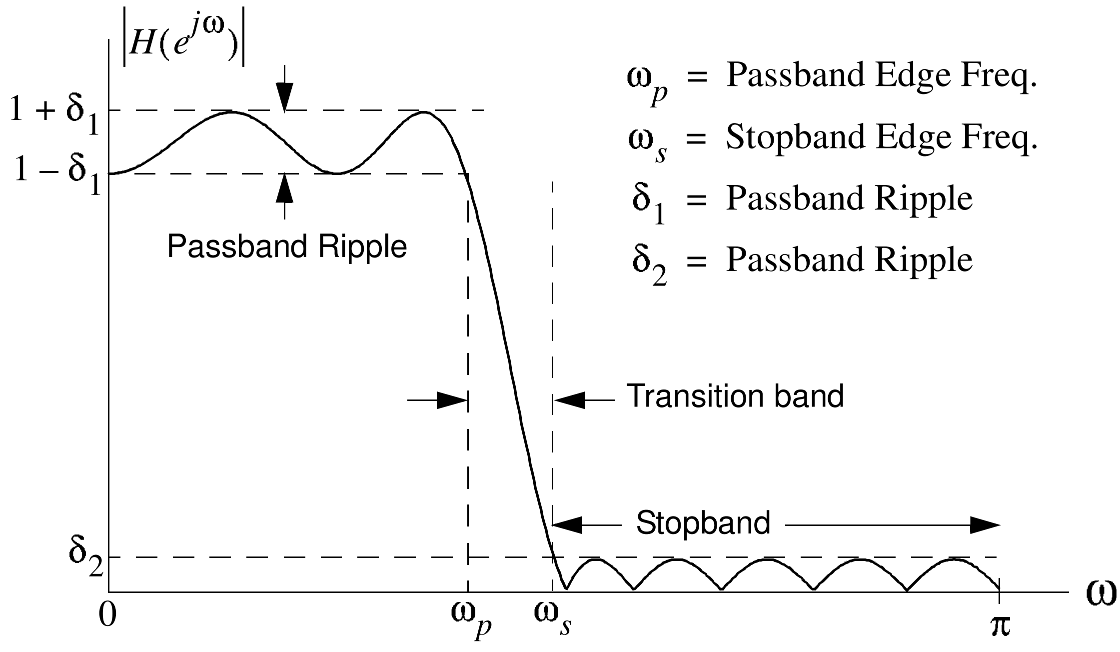
If $h[n]$ has finite energy and $h[n] = 0$ for $n < 0$, then

$$\int_{-\pi}^{\pi} |\ln|H(e^{j\omega})|| \, d\omega < \infty$$

- An important conclusion from the Paley-Wiener theorem is that $|H(e^{j\omega})|$ may be zero at a finite number of frequencies, but it cannot be zero over a band of frequencies without the integral becoming infinite $\Rightarrow$ Any ideal filter will be noncausal

- It is however possible to design realizable filters that approximate ideal filters

- Taking the lowpass filter as an example, the magnitude response of a realizable filter would be as follows



Magnitude response of a realizable lowpass filter

5.1.3 Phase Distortion and Group Delay

Consider an ideal delay system with impulse response

$$h_{id}[n] = \delta[n - n_d]$$

- The frequency response is

$$\begin{aligned} H_{id}(e^{j\omega}) &= e^{-j\omega n_d} \\ &= \underbrace{1}_{|H(e^{j\omega})|} \underbrace{\angle - \omega n_d}_{\angle H(e^{j\omega})}, \ |\omega| < \pi \end{aligned}$$

  assuming 2π periodicity

- Linear phase which results from a pure delay is considered an acceptable form of distortion in most cases

- For example, an ideal lowpass filter with linear phase can be defined as

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$$

  and the impulse response written as

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \ -\infty < n < \infty$$

- Note: An ideal lowpass filter is still unrealizable (noncausal) no matter how large we make n_d

A measure of the linearity of the phase response is the *group delay*.

- Consider the effect of the phase response on a narrowband signal

$$x[n] = s[n]\cos\omega_o n$$

where

$$X(e^{j\omega}) \approx 0, \quad |\omega - \omega_o| \geq \Delta\omega$$

- For ω near ω_o a linear approximation (two term Taylor series) to $\angle H(e^{j\omega})$ is

$$\angle H(e^{j\omega}) \approx -\phi_o - \omega n_d$$

- Thus we conclude that

$$y[n] = s[n - n_d]\cos(\omega_o n - \phi_o - \omega_o n_d)$$

which says that the time delay received by the envelope, $s[n]$, of $x[n]$, is given by the negative of the slope of the phase at ω_o

- Note that $\angle H(e^{j\omega})$ must be continuous around $\omega = \omega_o$ so the definition of group delay uses $\arg[H(e^{j\omega})]$ which as defined in Chapter 2 is continuous on $(0, \pi)$

- Formally group delay is defined as

$$\boxed{\tau(\omega) = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[H(e^{j\omega})]\}}$$

where the units are in samples

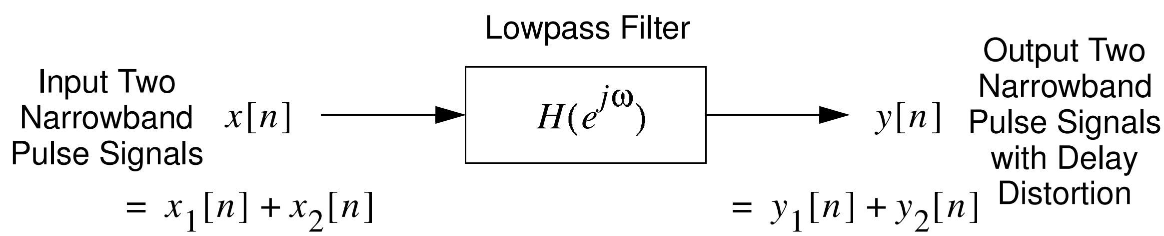
- The deviation of $\tau(\omega)$ from a constant indicates the degree of the nonlinearity of the phase

Example 5.1: **Delay Distortion on Narrowband Pulses**

- In this example we consider the impact of delay distortion, that is filtering signals with nonconstant group delay

- The case considered here is passing two narrowband pulse signals having different center frequency, through an LTI system with lowpass frequency response and nonconstant group delay

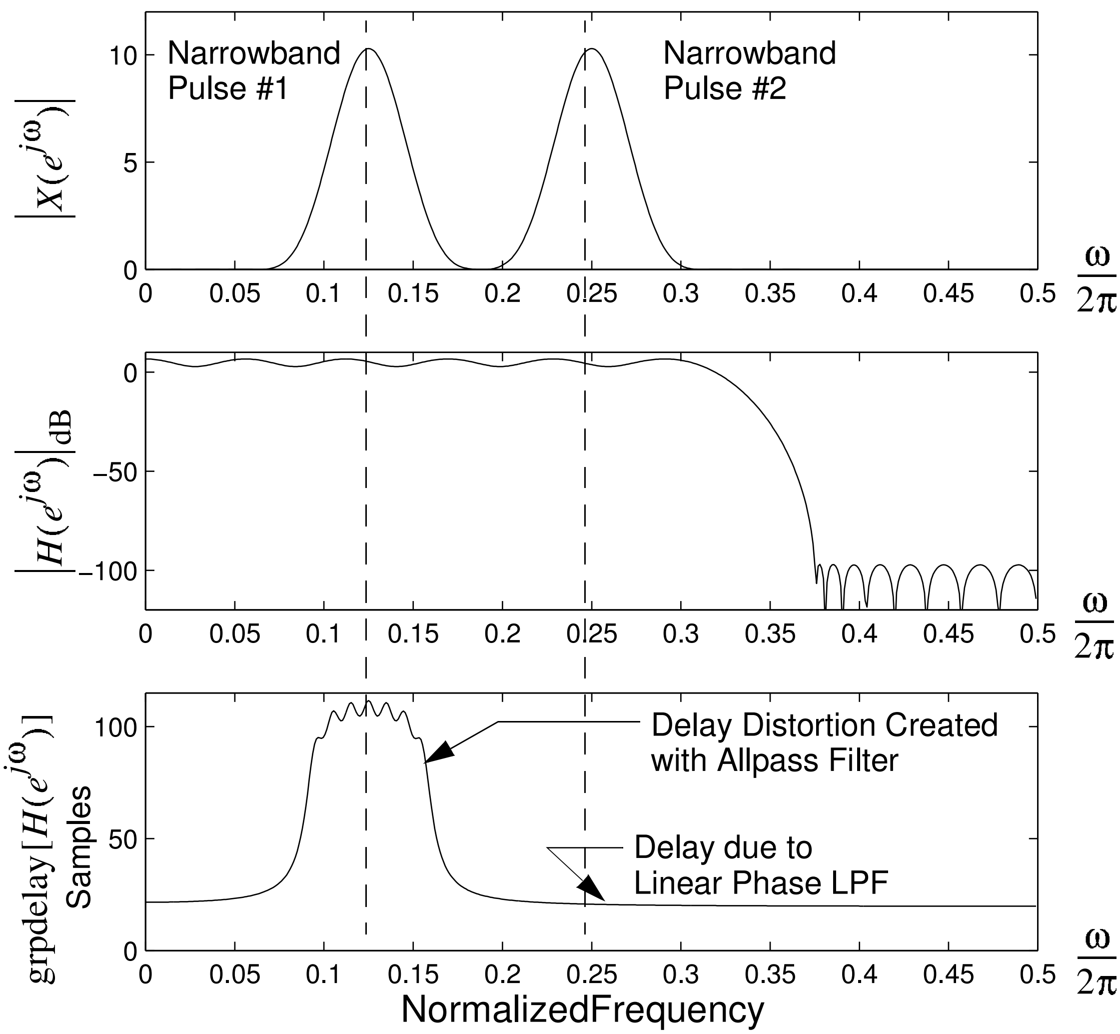
- The system block diagram is shown below



System block diagram

- The input signals $x_1[n]$ and $x_2[n]$ are created in MATLAB as short sinusoidal bursts of 50 samples weighted by a *Blackman* window function

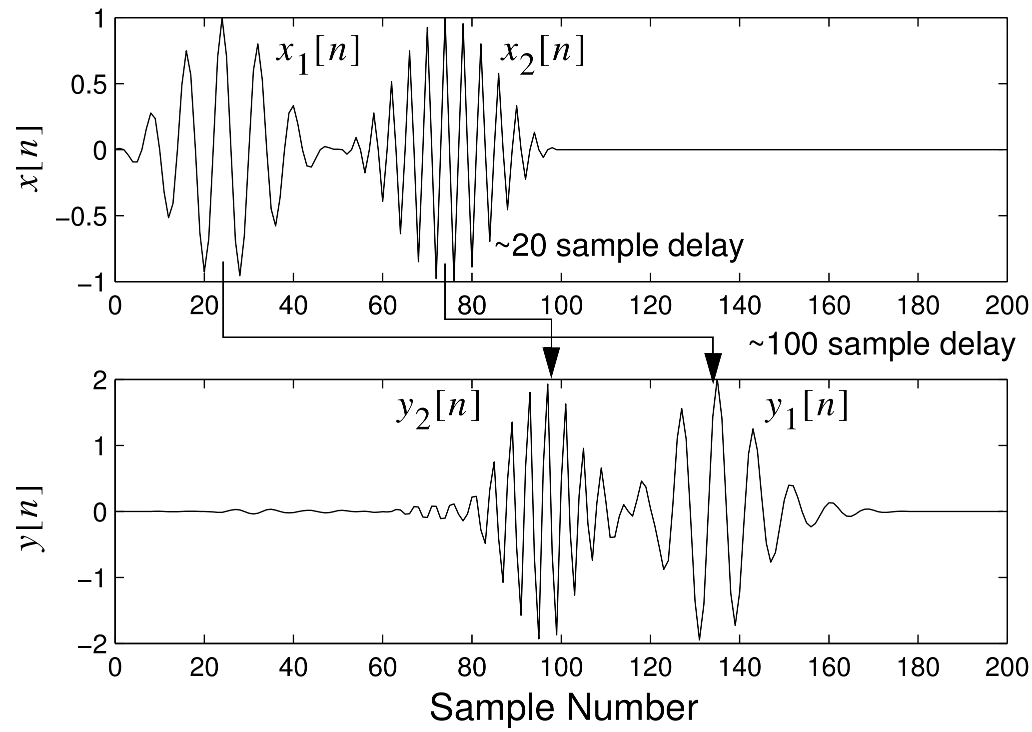
- The center frequency of the bursts is $\omega_1 = 2\pi \times 0.125$ and $\omega_2 = 2\pi \times 0.25$ respectively



Frequency domain plots: (a) Input spectra, (b) filter magnitude response, (c) filter group delay

- From the above figure we see that the group delay about ω_1 is $\sim$ 100 samples while the group delay about ω_2 is only $\sim$ 20 samples

- The narrowband pulses, both in the filter passband, will thus receive much different delays as they pass through the filter



Filter output time waveforms

- In this test case we see that the first pulse signal into the filter is the last signal to come out
- For a wideband pulse covering both the long and short portions of the group delay response, considerable pulse dispersion would occur

5.2 System Functions for Systems with LCCDE Representation

The class of systems with input/output relation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

and zero initial conditions is causal, linear, and time-invariant.

- If we take the z-transform of both sides using <u>only</u> linearity and time-invariance, we can write

$$
\begin{aligned}
H(z) &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \\
&= \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}
\end{aligned}
$$

- Each numerator factor $(1 - c_k z^{-1})$ contributes a zero at $z = c_k$ and a pole at $z = 0$

- Similarly each denominator factor $(1 - d_k z^{-1})$ contributes a pole at $z = d_k$ and a zero at $z = 0$

- If $H(z)$ is causal then $h[n]$ must be a right-sided sequence and thus the ROC of $H(z)$ must be the exterior of a circle with maximum pole radius

---

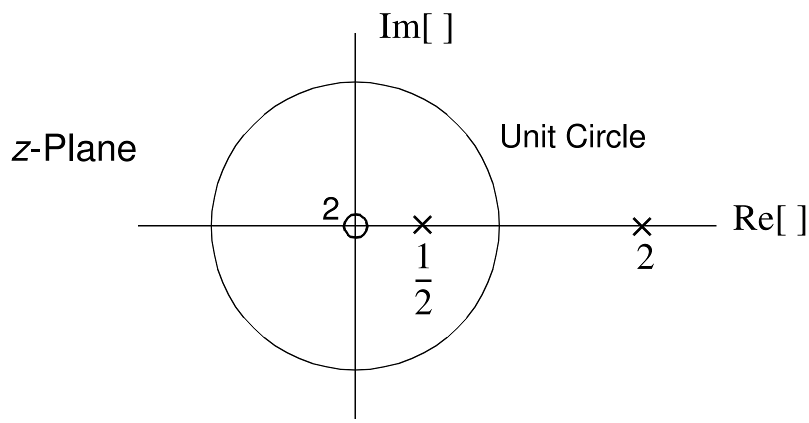
Example 5.2: **A Second-order LCCDE**

An LTI system has input/output relationship

$$
y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]
$$

- Taking the z-transform of both sides we get

$$
H(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 - 2z^{-1})}
$$



Pole-zero plot of $H(z)$

- There are three possible choices for the ROC:
  - If the system is to be causal $|z| > 2$, but then $H(z)$ is not stable
  - The system is stable if we choose $\frac{1}{2} < |z| < 2$, but then the system cannot be causal
  - If we choose $|z| < \frac{1}{2}$ the system is neither stable nor causal

5.2.1 Schür-Cohn Stability Test

When a system is causal we know that for stability the poles of $H(z)$ must lie inside the unit circle. Suppose

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

is the denominator polynomial of $H(z)$, then the roots of $A(z)$ must lie inside the unit circle.

- The Schür-Cohn stability test states that the polynomial $A(z)$ will have all of its roots inside the unit circle if and only if a set of coefficients known as *reflection coefficients* satisfy

$$|K_m| < 1 \ \ \forall \ \ m = 1, 2, \ldots, N$$

- The reflection coefficients can be calculated recursively by first defining the following polynomials

$$A_m(z) = \sum_{k=0}^{m} \alpha_m(k) z^{-k}, \quad \alpha_m(0) = 1$$

and the *reverse polynomial* $B_m(z)$

$$B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^{m} \alpha_m(m-k) z^{-k}$$

- To start the recursion set

$$A_N(z) = A(z) \ \text{ and } \ K_N = \alpha_N(N)$$

then for $m = N, N-1, N-2, \ldots, 1$ compute

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

where

$$K_m = \alpha_m(m)$$

- This procedure can be implemented on the computer such that only N^2 multiplications are required to obtain the K_m's

Example 5.3: Causal System Stability

Determine if the causal system

$$H(z) = \frac{1}{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}$$

is stable.

- For a simple second-order system the poles may be found directly
- For illustration purposes the Schür-Cohn Test will be used:

$$A_2(z) = 1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}$$

  and

$$K_2 = -\frac{1}{2}$$

- We now compute $A_1(z)$ and then K_1 using $B_2(z)$

$$B_2(z) = -\frac{1}{2} - \frac{7}{4}z^{-1} + z^{-2}$$

  thus

$$\begin{aligned} A_1(z) &= \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2} \\ &= \frac{4}{3}\left[\left(1 - \frac{1}{4}\right) - \left(\frac{7}{4} + \frac{7}{8}\right) z^{-1} + \left(-\frac{1}{2} + \frac{1}{2}\right) z^{-2}\right] \\ &= 1 - \frac{7}{2}z^{-1} \end{aligned}$$

- Finally $K_1 = -\frac{7}{2}$ which has magnitude greater than one, thus $H(z)$ is unstable

5.2.2 Inverse Systems

For any LTI system with system function $H(z)$, the corresponding system inverse, $H_i(z)$, is defined such that

$$H(z)H_i(z) = 1$$

thus

$$H_i(z) = \frac{1}{H(z)}$$

- The frequency response, $H_i(e^{j\omega})$, may not exist, e.g. consider an ideal lowpass filter; frequency components above ω_c are set to zero by $H(e^{j\omega})$ and thus cannot be recovered

- Systems with rational system functions have inverses of the form

$$\boxed{H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}}$$

  - Poles of $H(e^{j\omega}) \Rightarrow$ zeros of $H_i(e^{j\omega})$
  - Zeros of $H(e^{j\omega}) \Rightarrow$ poles of $H_i(e^{j\omega})$

- The ROC of $H_i(z)$ must overlap the ROC of $H(z)$, thus if $H(z)$ is causal the ROC of $H_i(z)$ must overlap $|z| > \max_k |d_k|$

- If $H(z)$ is also stable and we further require $H_i(z)$ to be stable, then both the poles and zeros of $H(z)$ must lie within the unit circle

Example 5.4: **A First-order System**

Let

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, \quad \text{ROC} = |z| > 0.9$$

- By definition

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$$

with ROC = $|z| > 0.5$ which overlaps with $|z| > 0.9$

- Inverse transforming we get

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$

which is both stable and causal

5.2.3 Impulse Response for Rational $H(z)$

First-Order Poles

A partial fraction expansion for rational $H(z)$, with only first-order poles can be written as

$$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- Inverse transforming term-by-term, assuming causality, yields

$$\boxed{h[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]}$$

- If $h[n]$ contains at least one term of the form $A_k(d_k)^n u[n]$, then the system is called an *infinite-impulse response* (IIR) system

No Poles Except at $z = 0$

- If $H(z)$ contains no poles, except at $z = 0$, then $N = 0$ and

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- The impulse response is simply

$$h[n] = \sum_{r=0}^{M} b_k \delta[n-k]$$

  and the system is called a *finite-impulse response* (FIR) system

- For FIR systems the difference equation can be represented as

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  which is identical to the convolution sum formula

5.3 Frequency Response for Rational System Functions

We are interested in the magnitude, phase, and group delay associated with $H(e^{j\omega})$.

- In terms of the poles and zeros of $H(z)$ we can write

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

- The magnitude response may be expressed as

$$|H(e^{j\omega})| = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} |1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N} |1 - d_k e^{-j\omega}|}$$

- Frequently we may wish to consider the *gain in dB* or *attenuation in dB* of $H(e^{j\omega})$

$$\text{Gain in db} = |H(e^{j\omega})|_{\text{dB}} = 20 \log_{10} |H(e^{j\omega})|$$

$$\text{Attenuation in dB} = \text{-Gain in dB}$$

- The phase response can be written as

$$\angle H(e^{j\omega}) = \angle \left[\frac{b_0}{a_0}\right] + \sum_{k=1}^{M} \angle[1 - c_k e^{-j\omega}] - \sum_{k=1}^{N} \angle[1 - d_k e^{-j\omega}]$$

- The principal value of the phase of $H(e^{j\omega})$ is denoted by $\text{ARG}[H(e^{j\omega})]$ where

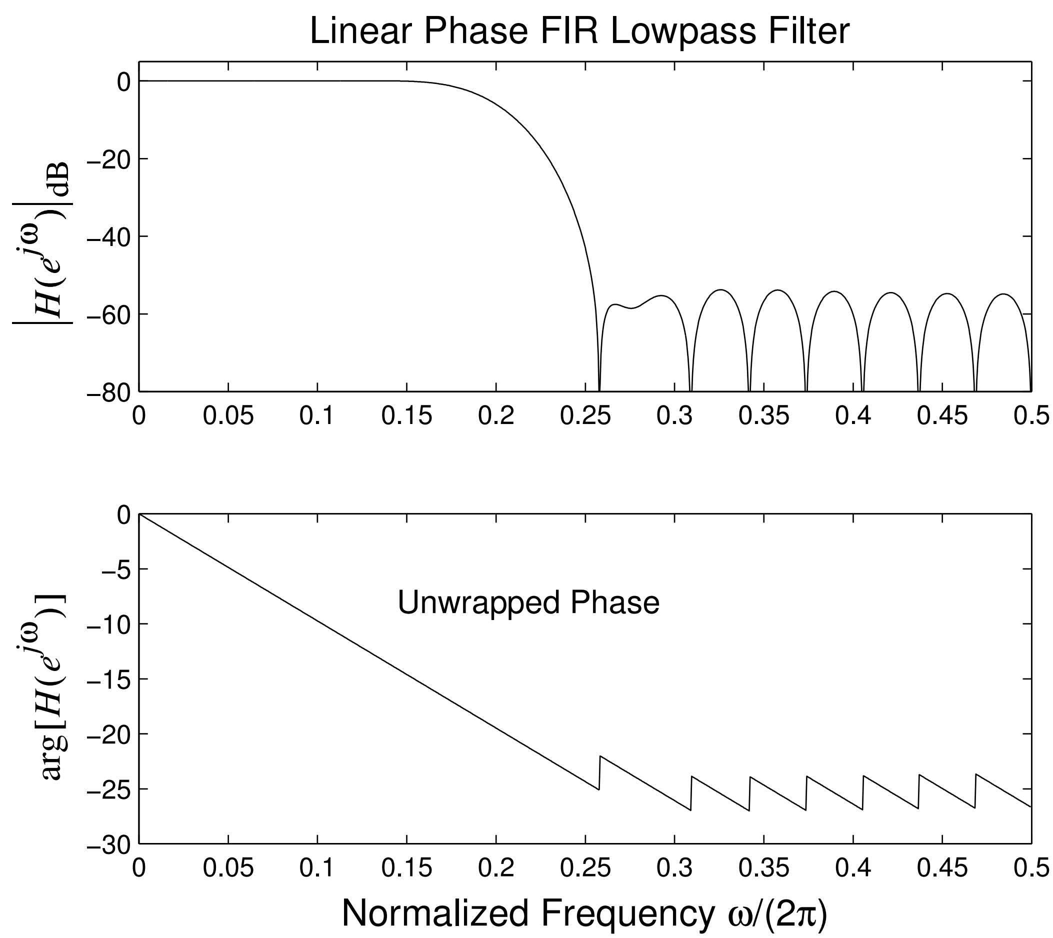
$$-\pi < \text{ARG}[H(e^{j\omega})] \le \pi$$

- In general $\text{ARG}[H(e^{j\omega})]$ contains $\pm 2\pi$ phase jumps, whereas $\arg[H(e^{j\omega})]$ is by definition continuous on $(0, \pi)$

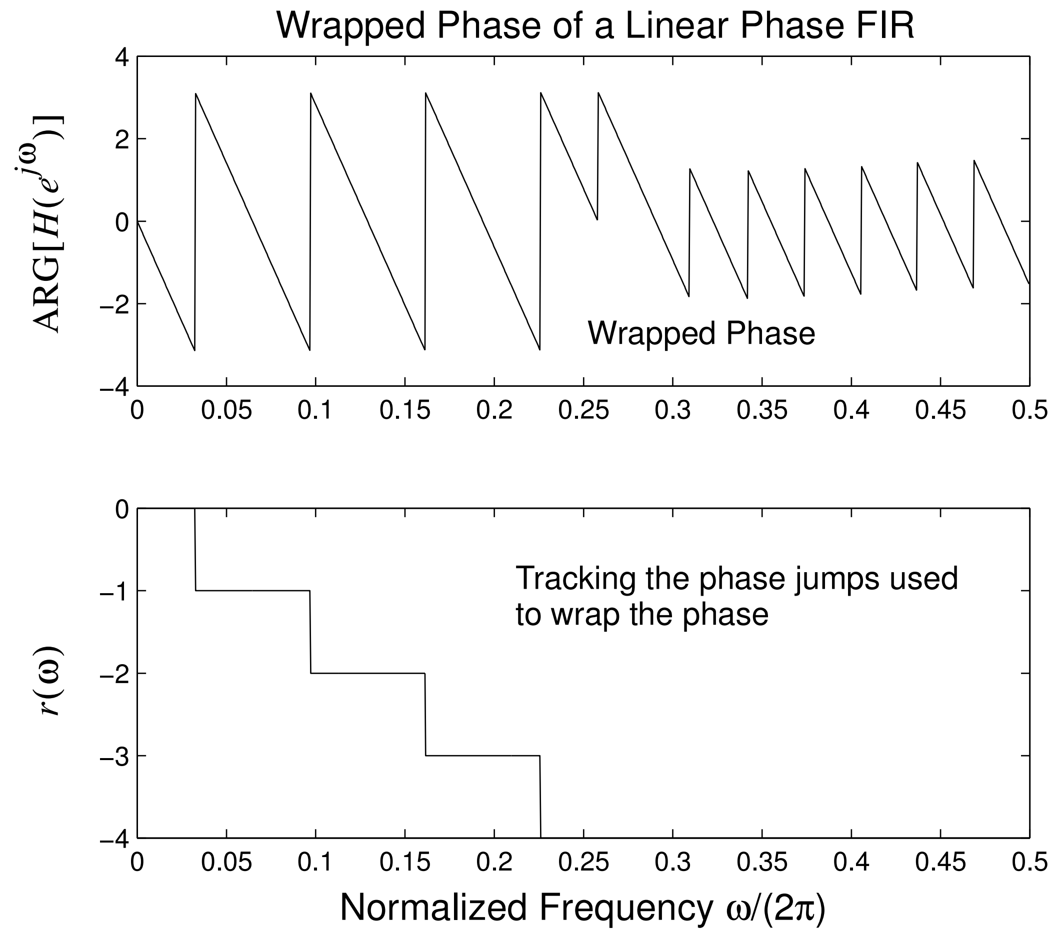
- To convert $\text{ARG}[H(e^{j\omega})]$ to $\arg[H(e^{j\omega})]$ we may write

$$\arg[H(e^{j\omega})] = \text{ARG}[H(e^{j\omega})] + 2\pi r(\omega)$$

  where $r(\omega)$ is an integer valued function



The magnitude and unwrapped ($\arg[H(e^{j\omega})]$) phase of a linear phase LPF



The corresponding wrapped phase (ARG[$H(e^{j\omega})$]) along with integer function $r(\omega)$ for the FIR LPF

- The group delay response is given by

$$\begin{aligned}\text{grd}[H(e^{j\omega})] &= -\sum_{k=1}^{M} \frac{d}{d\omega}(\arg[1 - c_k e^{-j\omega}]) \\ &\quad + \sum_{k=1}^{N} \frac{d}{d\omega}(\arg[1 - d_k e^{-j\omega}])\end{aligned}$$

- Strictly speaking the group delay is the derivative of $\arg[H(e^{j\omega})]$, but $\text{ARG}[H(e^{j\omega})]$ may be used if the impulses caused by the $\pm 2\pi$ discontinuities are ignored

5.3.1 Geometric Evaluation of $H(e^{j\omega})$

- To begin with consider

$$H(z) = \left(\frac{b_0}{a_0}\right) z^{N-M} \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{k=1}^{N}(z - d_k)}$$

- Now substitute $z \to e^{j\omega}$ and write $|H(e^{j\omega})|$ as a ratio of vector magnitudes

$$\begin{aligned} |H(e^{j\omega})| &= \left(\frac{b_0}{a_0}\right) |e^{j\omega}|^{N-M} \frac{\prod_{k=1}^{M} |e^{j\omega} - c_k|}{\prod_{k=1}^{N} |e^{j\omega} - d_k|} \\ &= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M} |\vec{v}_{ck}|}{\prod_{k=1}^{N} |\vec{v}_{dk}|} \end{aligned}$$

  where $\vec{v}_{ck} = e^{j\omega} - c_k$ and $\vec{v}_{dk} = e^{j\omega} - d_k$

- Similarly

$$\angle H(e^{j\omega}) = \sum_{k=1}^{M} \angle(\vec{v}_{ck}) - \sum_{k=1}^{N} \angle(\vec{v}_{dk}) + (N - M)\omega$$

Example 5.5: **A Single Complex Zero**

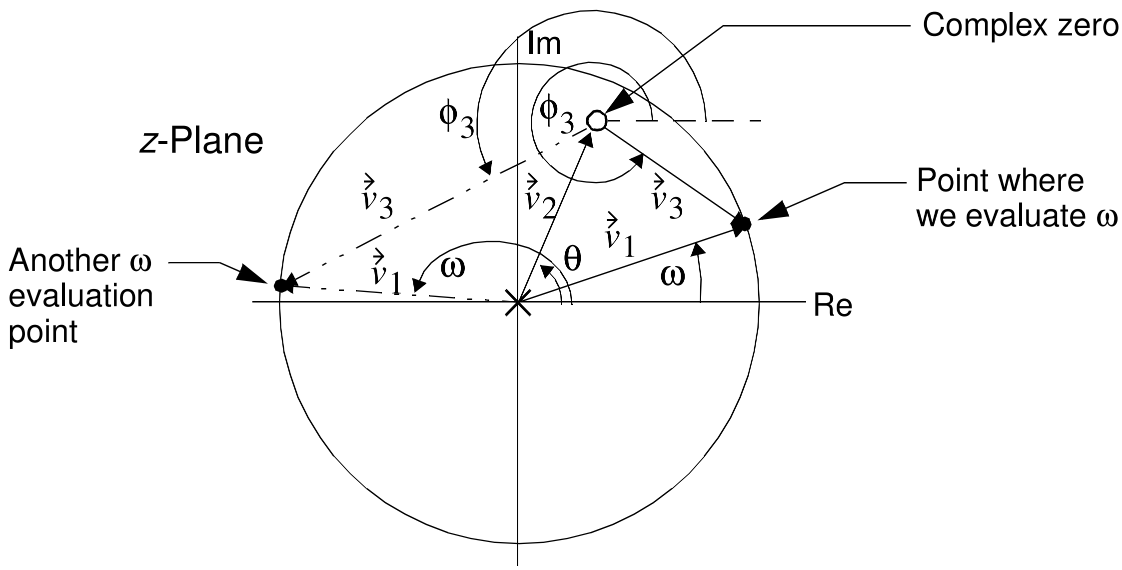
Consider the frequency response of a single complex zero

$$H(z) = 1 - re^{j\theta}z^{-1} = \frac{z - re^{j\theta}}{z},\ r < 1$$

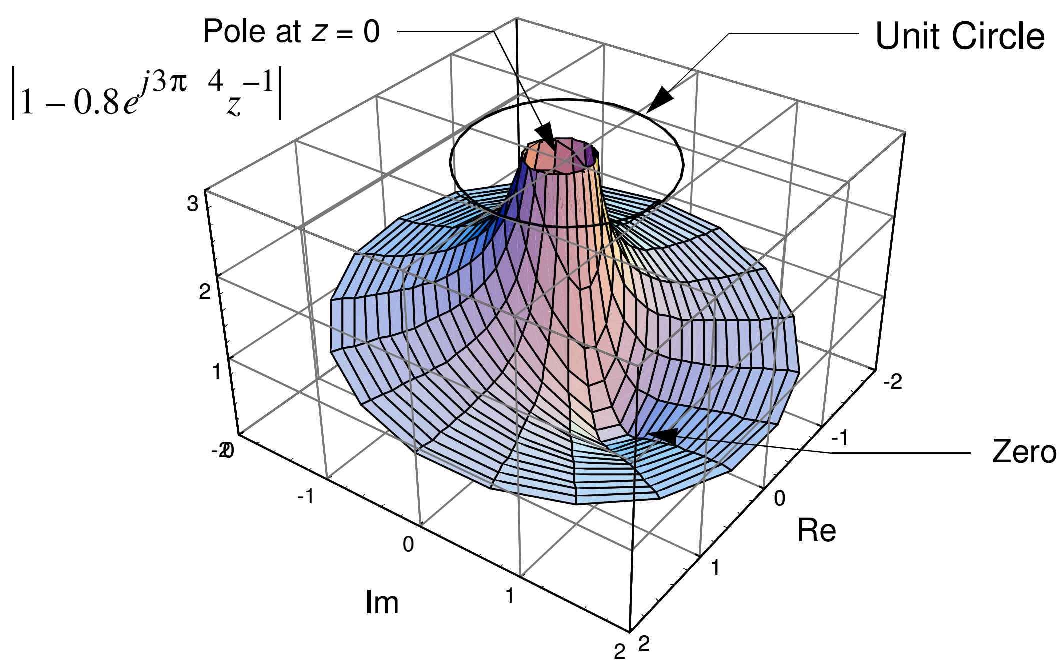
where here $c_1 = re^{j\theta}$

- Writing $|H(e^{j\omega})|$ as a ratio of vector magnitudes we have

$$|H(e^{j\omega})| = \frac{|e^{j\omega} - re^{j\theta}|}{|e^{j\omega}|} \overset{\text{in text}}{=} \frac{|\vec{v}_3|}{|\vec{v}_1|}$$



z-plane vectors for a single zero system



Surface plot of $|H(z)|$

- Analytically the gain in dB is

$$|H(e^{j\omega})|_{\text{db}} = 10 \log_{10}[1 + r^2 - 2r\cos(\omega - \theta)]$$

- The corresponding phase is

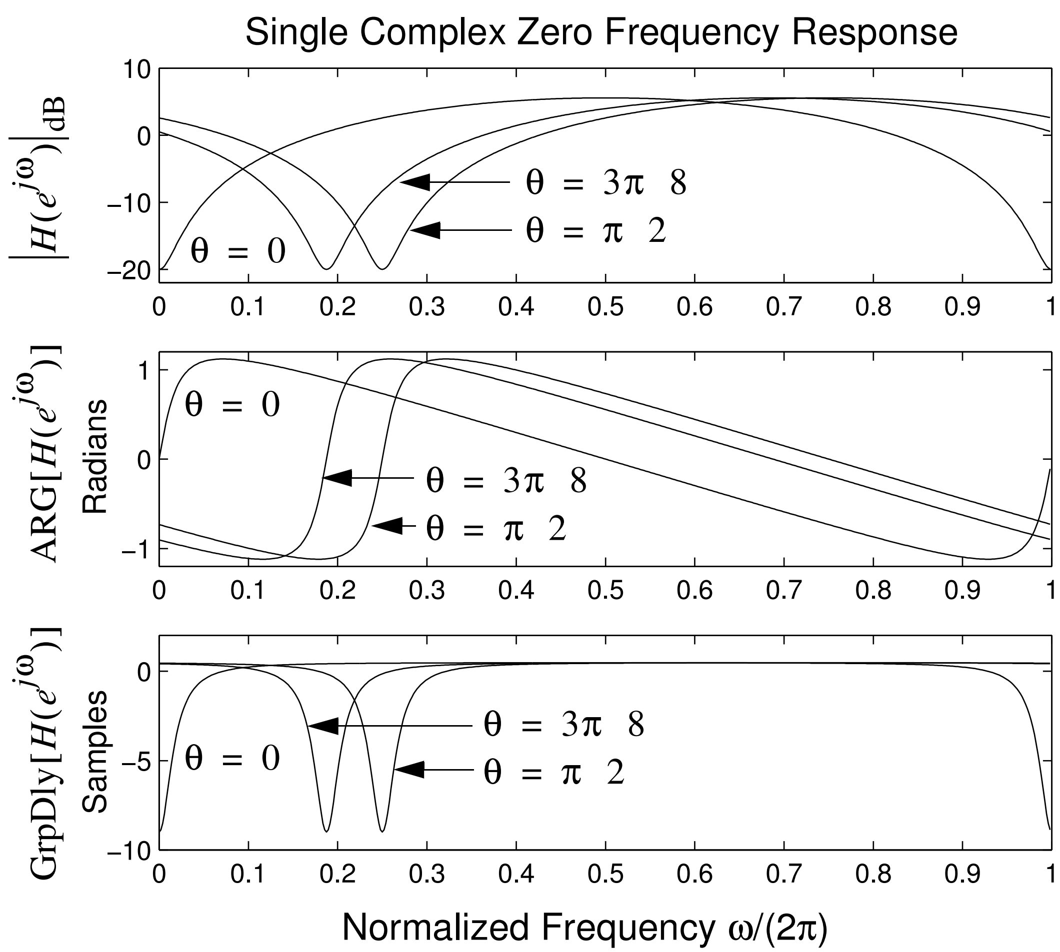
$$\angle H(e^{j\omega}) = \angle(e^{j\omega} - re^{j\theta}) - \omega \overset{\text{in text}}{=} \underbrace{\angle(\vec{v}_3)}_{\phi_3} - \omega$$

- By direct calculation

$$\text{ARG}[H(e^{j\omega})] = \tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

- We can also calculate the group delay to be

$$\text{grd}[H(e^{j\omega})] = \frac{r^2 - r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}$$

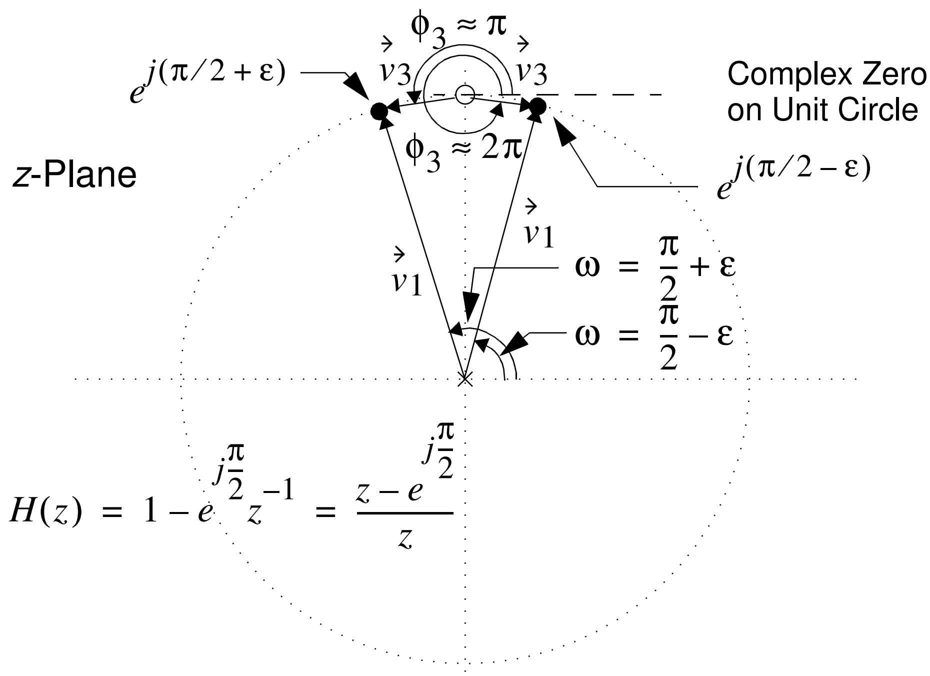


Frequency response plots for a single zero system with $r = 0.9$ and three values of theta. (a) gain in dB. (b) phase in radians. (c) group delay in samples

- Note: If the system were a single pole instead, then all of the response curves shown would have a sign reversal

A Zero on the Unit Circle

- Consider now a zero located at $z_1 = e^{j\theta}$, in particular $\theta = \pi/2$



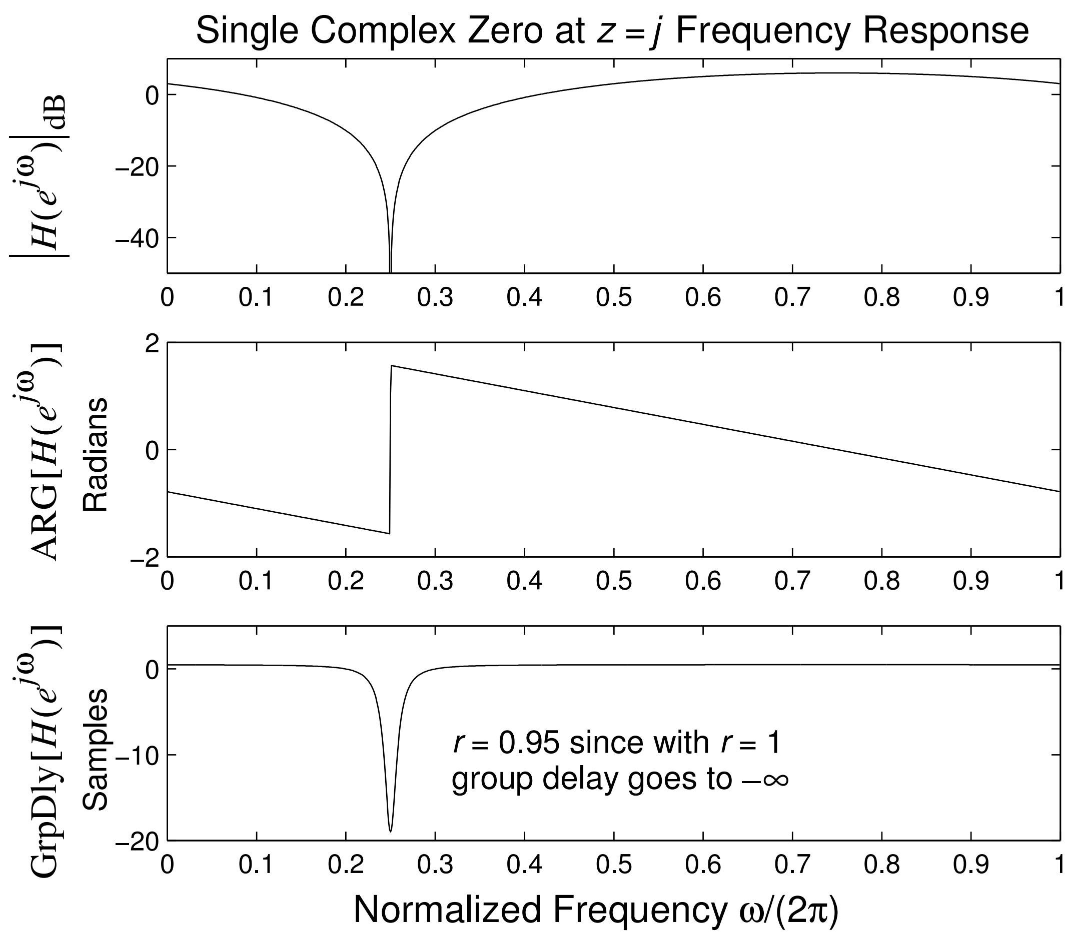
z-plane vectors for a single zero at $z = e^{j\pi/2}$ and ω near $\pi/2$

- The magnitude response dips sharply to zero ($-\infty$ dB) when $\omega = \pi/2$

- By considering the phase response either side of $\pi/2$ we see from the figure that the phase jumps by π radians as ω passes through the zero angle

$$\angle H(e^{j(\pi/2-\epsilon)}) \simeq 2\pi - (\pi/2 - \epsilon) = 3\pi/2 + \epsilon$$
$$\angle H(e^{j(\pi/2+\epsilon)}) \simeq \pi - (\pi/2 + \epsilon) = \pi/2 - \epsilon$$

- The group delay is not defined at $\omega = \pi/2$ due to the phase jump



Frequency response plots for a single zero at $z = e^{j\pi/2}$. (a) gain in dB. (b) phase in radians. (c) group delay in samples

A Zero Outside the Unit Circle

- If $r > 1$ we see that the magnitude response is similar to the $r < 1$ case

- The phase response however picks up a 2π jump when ω passes through the angle θ
- The group delay response curve has a positive dip, the opposite of the $r < 1$ case

Example 5.6: **Conjugate Pole Pair**

Find the frequency response of a conjugate pole pair

$$\begin{aligned} H(z) &= \frac{1}{(1 - re^{j\theta}z^{-1})(1 - re^{-j\theta}z^{-1})} \\ &= \frac{1}{1 - 2r\cos\theta\, z^{-1} + r^2 z^{-2}} \end{aligned}$$

where here $d_1 = re^{j\theta}$ and $d_2 = re^{-j\theta}$.

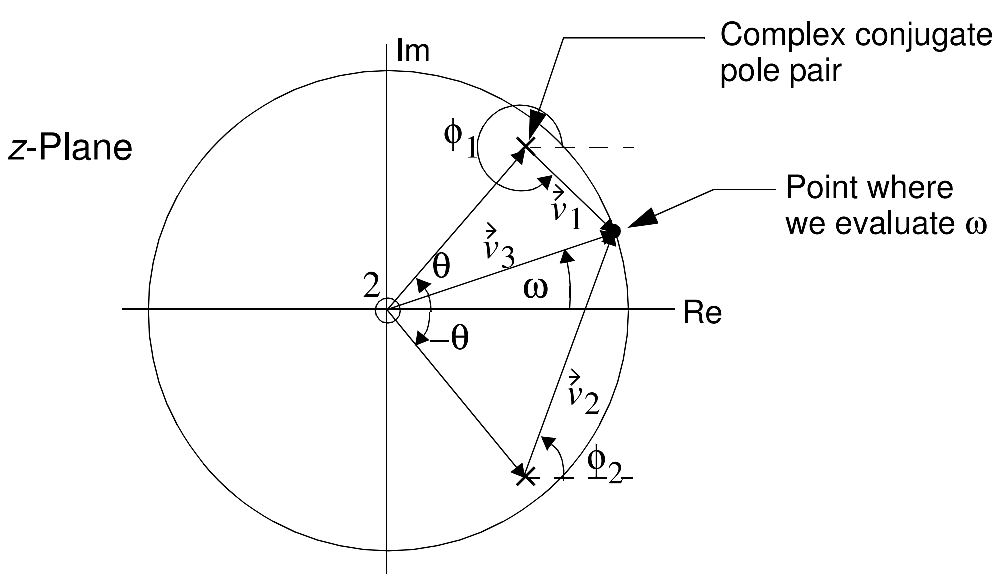
- Writing $|H(e^{j\omega})|$ as a ratio of vector magnitudes gives

$$|H(e^{j\omega})| = \frac{1}{|\vec{v}_1||\vec{v}_2|}$$

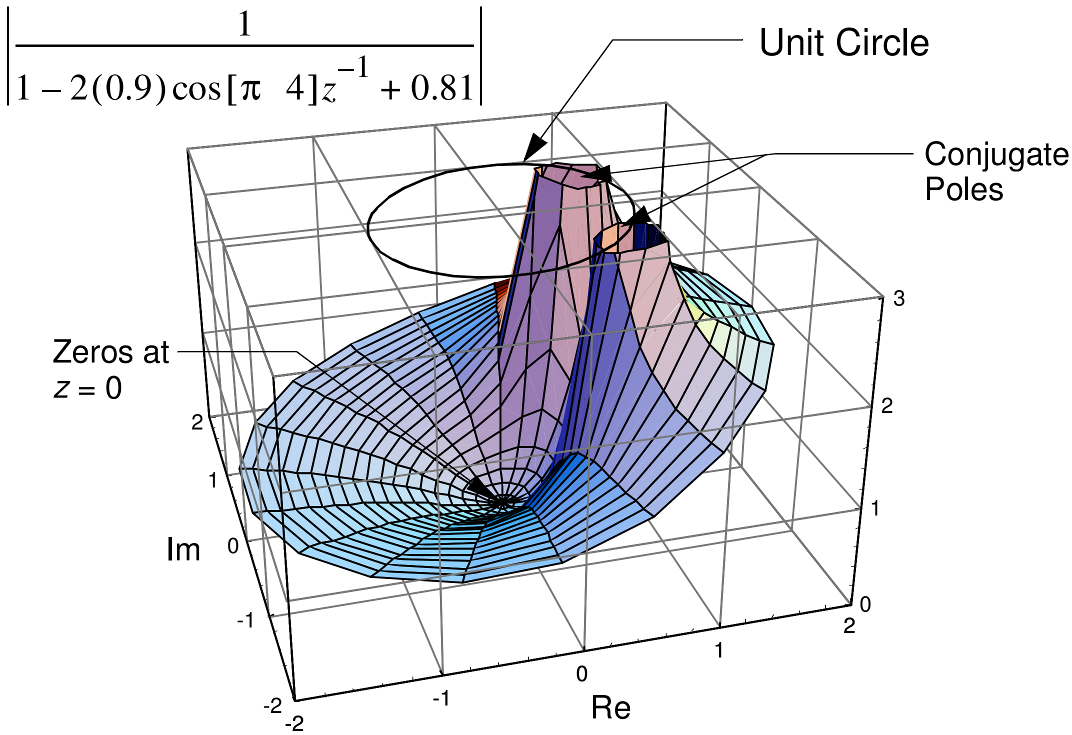
where $\vec{v}_1 = e^{j\omega} - re^{j\theta}$ and $\vec{v}_2 = e^{j\omega} - re^{-j\theta}$

- The phase in terms of vector angles is

$$\angle H(e^{j\omega}) = 2\omega - \angle\vec{v}_1 - \angle\vec{v}_2 = 2\omega - \phi_1 - \phi_2$$



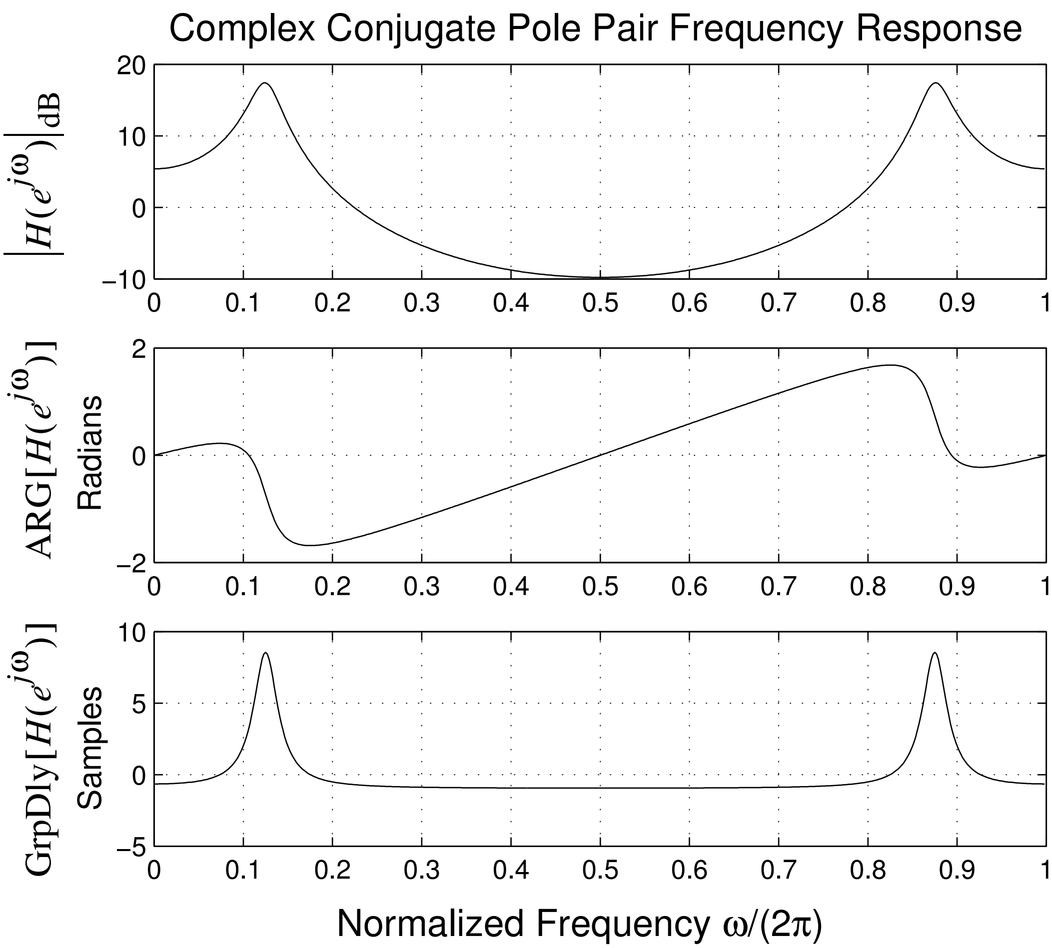
z-plane vectors for a conjugate pole pair system



Surface plot of $|H(z)|$

- By direct calculation we also obtain the quantities

$$
\begin{aligned}
|H(e^{j\omega})|_{\text{db}} &= -10\log_{10}[1 + r^2 - 2r\cos(\omega-\theta)] \\
&\quad - 10\log_{10}[1 + r^2 - 2r\cos(\omega+\theta)] \\
\text{ARG}[H(e^{j\omega})] &= -\tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] \\
&\quad - \tan^{-1}\left[\frac{r\sin(\omega+\theta)}{1-r\cos(\omega+\theta)}\right] \\
\text{grd}[H(e^{j\omega})] &= -\frac{r^2 - r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)} \\
&\quad - \frac{r^2 - r\cos(\omega+\theta)}{1+r^2-2r\cos(\omega+\theta)}
\end{aligned}
$$



Frequency response plots for a conjugate pole pair system with $r = 0.9$ and $\theta = \pi/4$. (a) gain in dB. (b) phase. (c) group delay

Example 5.7: **Moving Average or Comb FIR Filter**

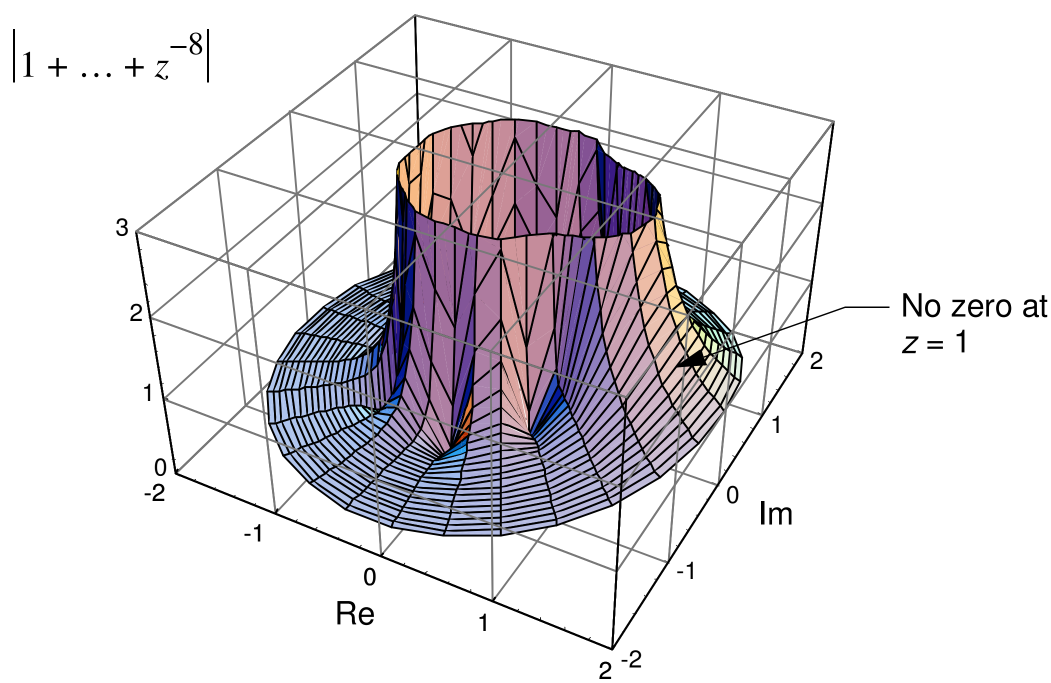
- Consider an FIR system of the form

$$h[n] = \sum_{k=0}^{7} \delta[n-k]$$

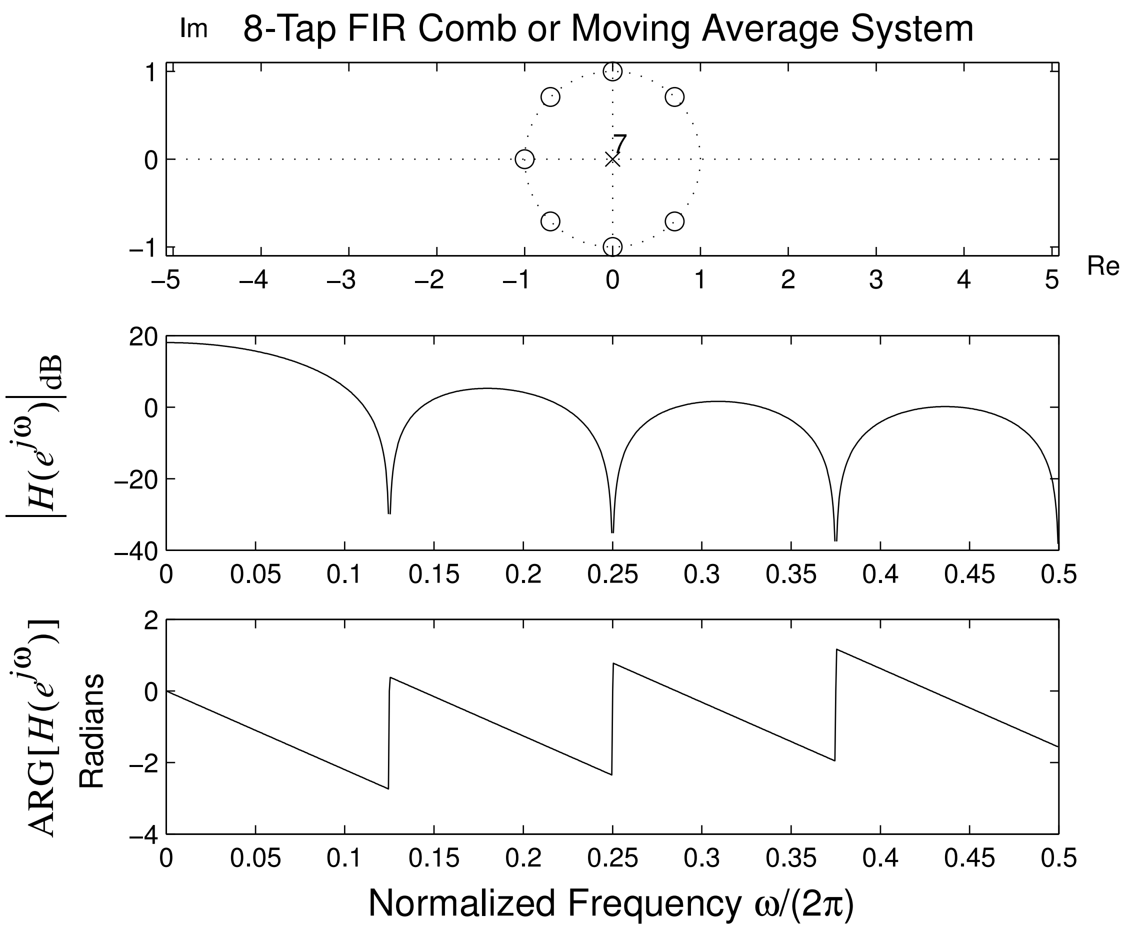
which has system function

$$H(z) = \sum_{n=0}^{7} z^{-n} = \frac{1 - z^{-8}}{1 - z^{-1}}$$

- The pole at $z = 1$ cancels with a zero at $z = 1$ giving a lowpass or *comb* filter frequency response with transmission zeros at locations $\omega = 2\pi k/8 = \pi k/4$, $k = 1, \ldots, 7$



Surface plot of $|H(z)|$



Frequency response plots for a 8-tap moving average FIR system; (a) pole-zero map (b) gain in dB. (c) phase.

Example 5.8: Second-Order Bandpass Resonator

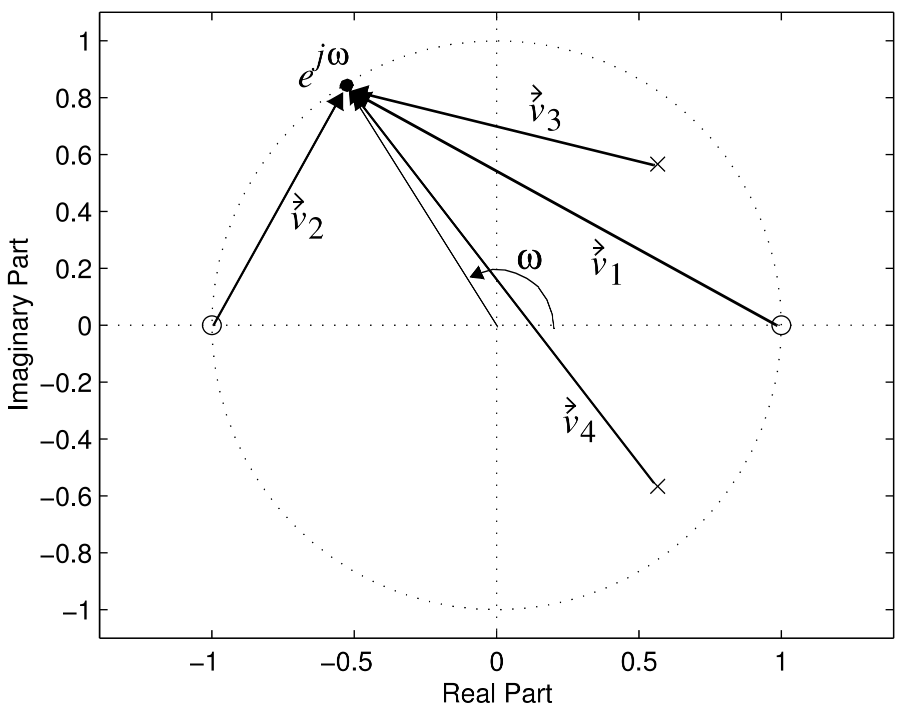
- Consider a second-order *bandpass resonator* with $H(z)$ given by

$$\begin{aligned} H(z) &= \frac{b_0(1 - z^{-2})}{1 + a_1 z^{-1} + a_2 z^{-2}} \\ &= \frac{b_0(z+1)(z-1)}{(z-p_1)(z-p_2)} \end{aligned}$$

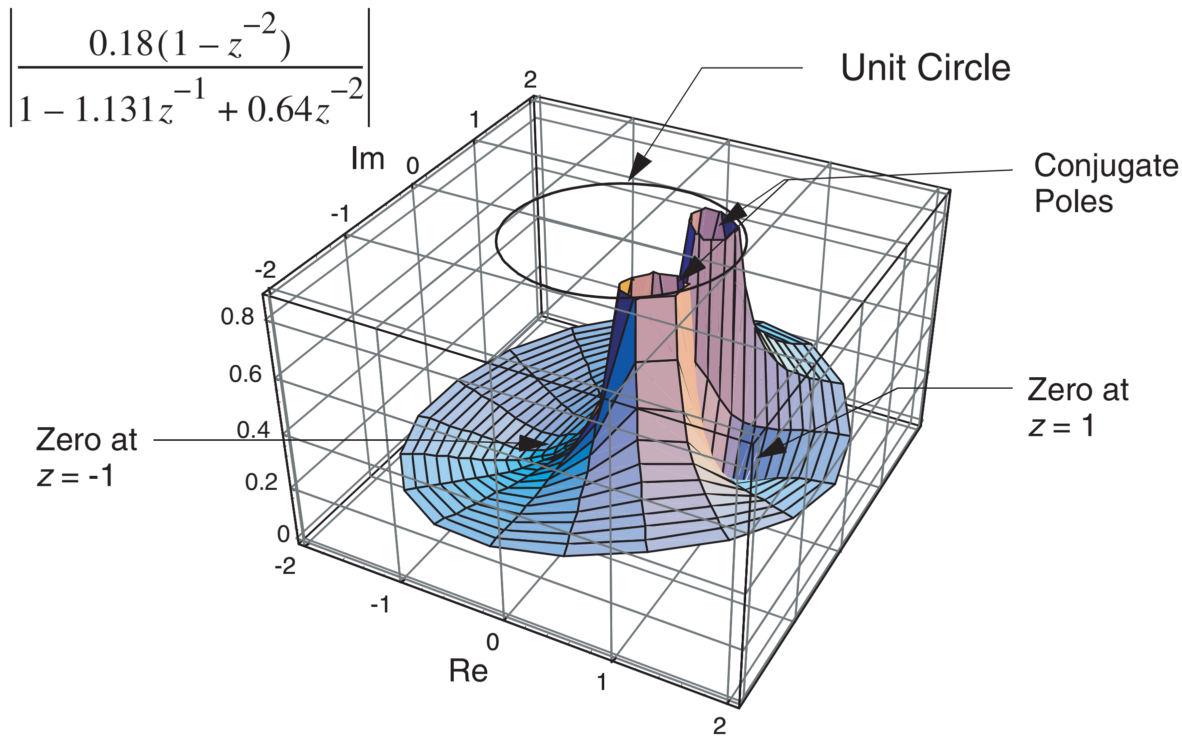
where $p_1, p_2 = -a_1/2 \pm j\sqrt{a_2 - a_1^2/4}$

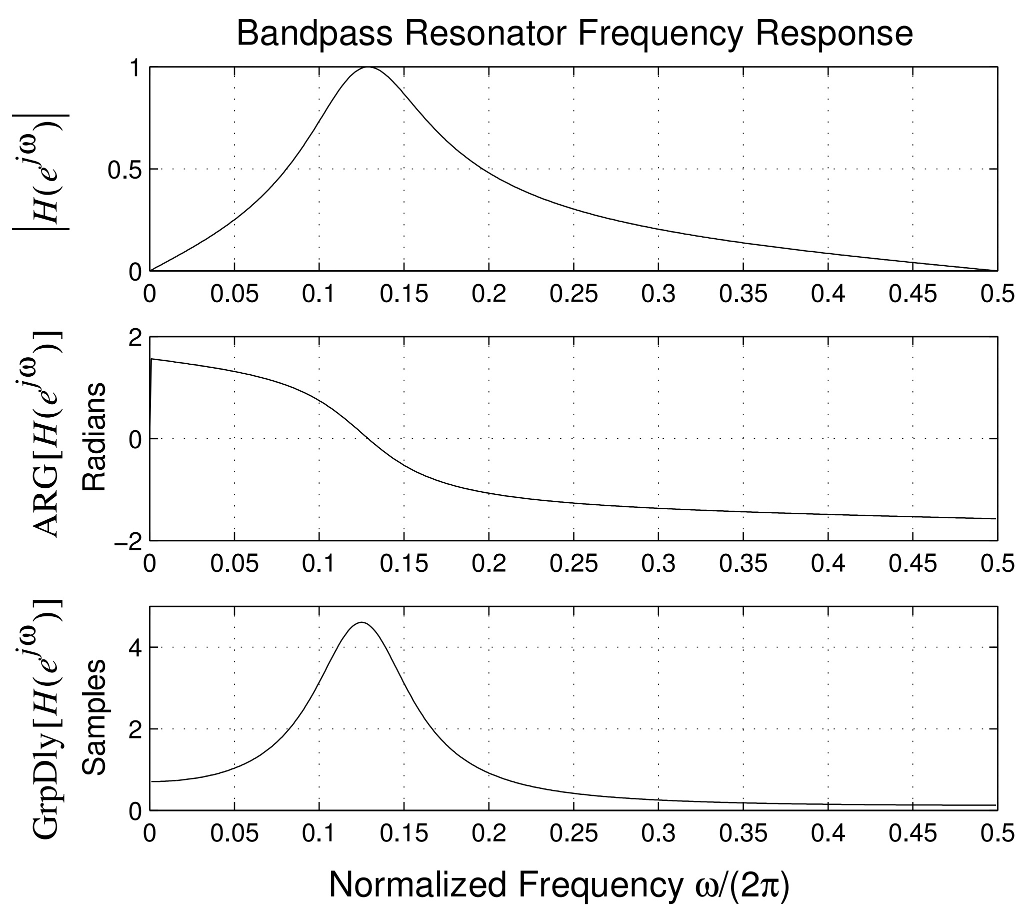
- Let $a_1 = -1.131$, $a_2 = 0.64$,and $b_0 = 0.18$ then,

$$p_1, p_2 = 0.556 \pm j0.566 = 0.8\angle \pm 45^\circ$$



Pole-zero map from MATLAB with vectors overlaid


$\left|\dfrac{0.18(1-z^{-2})}{1-1.131z^{-1}+0.64z^{-2}}\right|$
Unit Circle
Conjugate Poles
Zero at $z = 1$
Zero at $z = -1$
Im
Re
2
1
0
-1
-2
0.8
0.6
0.4
0.2
0


Bandpass Resonator Frequency Response
$|H(e^{j\omega})|$
ARG[$H(e^{j\omega})$] Radians
GrpDly[$H(e^{j\omega})$] Samples
1
0.5
0
2
−2
4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Normalized Frequency ω/(2π)

5.4 The Relationship Between Magnitude and Phase

For systems described by LCCDEs there is a constraint between the magnitude and phase:

Magnitude response known and the number of poles and zeros known	$\Longrightarrow$	A finite number of choices for the phase
Phase response known and the number of poles and zeros known	$\Longrightarrow$	To within a scale factor a finite number of choices for the magnitude

- To begin the investigation consider $|H(e^{j\omega})|^2$

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= H(z)H^*(1/z^*)\big|_{z=e^{j\omega}} \end{aligned}$$

- If $H(z)$ is rational we can write

$$\begin{aligned} C(z) &= H(z)H^*(1/z^*) \\ &= \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}(1-c_k z^{-1})(1-c_k^* z)}{\prod_{k=1}^{N}(1-d_k z^{-1})(1-d_k^* z)} \end{aligned}$$

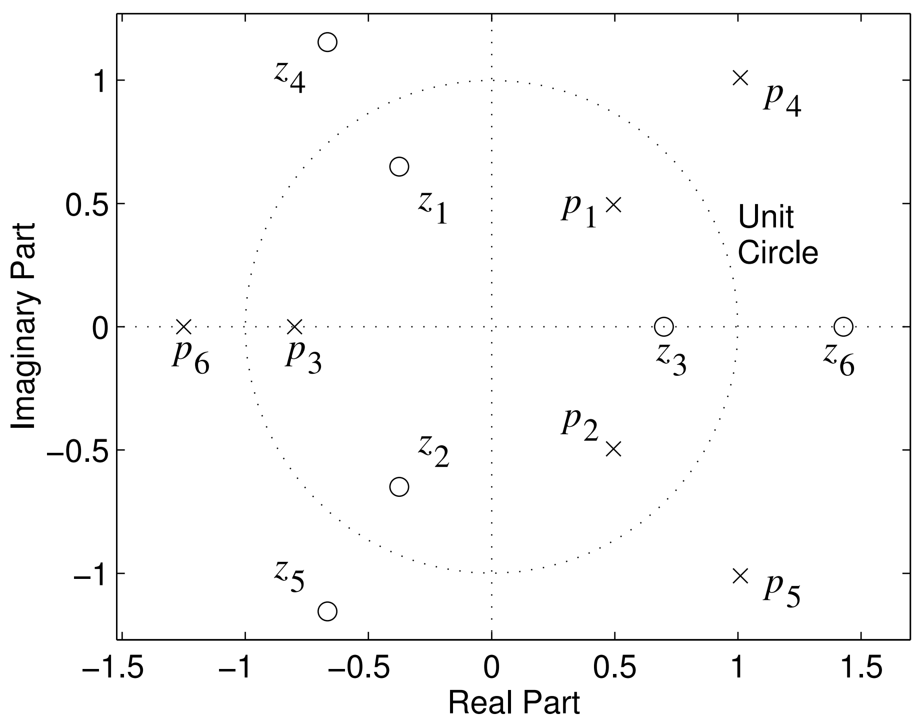
- Note that the poles and zeros of $C(z)$ occur in conjugate reciprocal pairs
  - If one pole (zero) of each pair is inside the unit circle, then the other is outside the unit circle

- If one pole (zero) of each pair is on the unit circle, the other must also be on the unit circle

- If $H(z)$ is causal <u>and</u> stable, then the poles of $H(z)$ must lie inside the unit circle
  - The poles of $H(z)$ can be identified from the poles of $C(z)$
  - The zeros however are not known uniquely from $C(z)$

Example 5.9: A 3-Pole/3-Zero System

Suppose we are given the following pole-zero plot of $C(z)$



Pole-zero plot of $C(z)$

- If $H(z)$ is causal and stable, then the poles of $H(z)$ must be p_1, p_2, and p_3

- The zeros of $H(z)$, assuming real coefficients, can be one of four possibilities:
  - case 1: $z_1,\ z_2$, and z_3 (min. phase)
  - case 2: $z_1,\ z_2$, and z_6
  - case 3: $z_4,\ z_5$, and z_3
  - case 4: $z_4,\ z_5$, and z_6 (max. phase)
- For all four cases the magnitude response of $H(e^{j\omega})$ is identical
- If $H(z)$ is not restricted to three poles and three zeros, then there are an infinite number of choices for $H(z)$ from the given $C(z)$
- To demonstrate this let $H(z)$ contain a factor of the form
$$\frac{z^{-1} - a^*}{1 - az^{-1}} = \frac{-a^*(1 - (1/a^*)z^{-1})}{1 - az^{-1}},$$
then $C(z)$ is unchanged, but the phase of $H(z)$ is modified
- Why? On the unit circle we have
$$\left|\frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}}\right| = |e^{-j\omega}|\frac{|1 - a^*e^{j\omega}|}{|1 - ae^{-j\omega}|} = 1$$
- These factors are known as *allpass factors* since they have unity magnitude on the unit circle
- Any number of allpass factors may be present in $H(z)$, but they cannot be identified from $C(z)$ i.e.
$$\frac{z^{-1} - a^*}{1 - az^{-1}} \cdot \frac{z - a}{1 - a^*z} = 1$$

5.5 Allpass Systems

A system is *allpass* if

$$|H_{\text{ap}}(e^{j\omega})| = 1, \quad \forall\ \omega$$

- A first-order allpass system is of the form

$$H_{\text{ap}}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

- We have already seen that $H_{\text{ap}}(e^{j\omega})$ has unity magnitude

- If $h_{\text{ap}}[n]$ is in general assumed to be complex, then the general form of $H_{\text{ap}}(z)$ is

$$\boxed{\begin{aligned} H_{\text{ap}}(z) &= \frac{z^{-N} + a_1^* z^{-N+1} + \cdots + a_N^*}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\ &= z^{-N}\frac{D^*(z^{-1})}{D(z)} \end{aligned}}$$

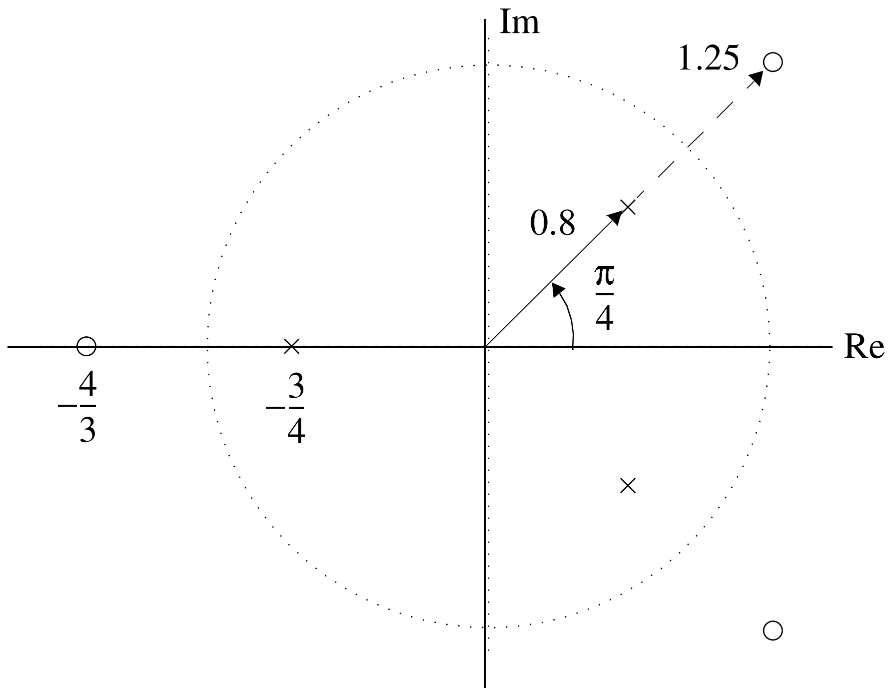
  where the a_k's are in general complex coefficients

- To show that the above is allpass we simply write

$$|H_{\text{ap}}(e^{j\omega})| = |e^{-j\omega}|^N \left|\frac{D^*(e^{-j\omega})}{D(e^{j\omega})}\right| = 1$$

  since $[D^*(e^{-j\omega})]^* = D(e^{j\omega})$ the magnitudes are equal

- For $H_{\text{ap}}(z)$ causal and stable the poles of $H_{\text{ap}}(z)$ must be inside the unit circle, therefore the zeros must be outside the unit circle e.g. see the following figure



- If $H_{\text{ap}}(z)$ is causal and stable, then the group delay contributed by $H_{\text{ap}}(z)$ is always positive and the phase is nonpositive

  proof:

$$\arg[H_{\text{ap}}(e^{j\omega})] = -\int_0^{\omega} \text{grd}[H_{\text{ap}}(e^{j\phi})]\, d\phi + \arg[H_{\text{ap}}(e^{j0})]$$

  and

$$H_{\text{ap}}(e^{j0}) = \frac{D(1)}{D(1)} = 1 \rightarrow \arg[H_{\text{ap}}(e^{j0})] = 0$$

  Now since $\text{grd}[H_{\text{ap}}(e^{j\omega})] \geq 0$ the integrand is nonnegative, thus making

$$\arg[H_{\text{ap}}(e^{j\omega})] \leq 0 \quad \text{for } 0 \leq \omega < \pi$$

- An important application of allpass systems is as group delay compensators

- For the one-pole all-pass system the group delay is given by

$$\operatorname{grd}\left[\frac{e^{j\omega} - re^{j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)}$$
$$= \frac{1 - r^2}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

  which is positive if $r < 1$

5.6 Minimum-Phase Systems

An LTI system $H(z)$ is *minimum-phase* if it is causal and stable and its inverse $1/H(z)$ is also causal and stable

- The definition of minimum-phase thus requires that both the poles and zeros of $H(z)$ be inside the unit circle

- If $H(z)$ has zeros on the unit circle, then by definition the system is nonminimum-phase, however many of the minimum-phase system properties still hold

- Observe that <u>any</u> $H(z)$ can be written as a cascade

$$H(z) = H_{\min}(z)H_{\text{ap}}(z)$$

  where $H_{\min}(z)$ is a minimum-phase system and $H_{\text{ap}}(z)$ is an allpass system

- A nonminimum-phase system can always be made into a minimum-phase system by simply reflecting all the zeros outside the unit circle to their conjugate reciprocal locations inside the unit circle. The converse is also possible.

- To see how this works consider

$$H(z) = H_1(z)(z^{-1} - c^*)$$

where $z = 1/c^*$, $|c| < 1$ is the only zero of $H(z)$ outside the unit circle and $H_1(z)$ is minimum phase

- We can find a minimum phase equivalent system for $H(z)$ by writing

$$H(z) = H_1(z)(1 - cz^{-1})\frac{z^{-1} - c^*}{1 - cz^{-1}}$$

- The term $H_1(z)(1 - cz^{-1})$ is minimum phase and is related to $H(z)$ via the all-pass section $(z^{-1} - c^*)/(1 - cz^{-1})$

- In any case the frequency response magnitudes are identical

---

Example 5.10: Minimum-Phase with All-Pass Decomposition

- Consider the causal stable system

$$H(z) = \frac{\left(1 - \frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}}$$

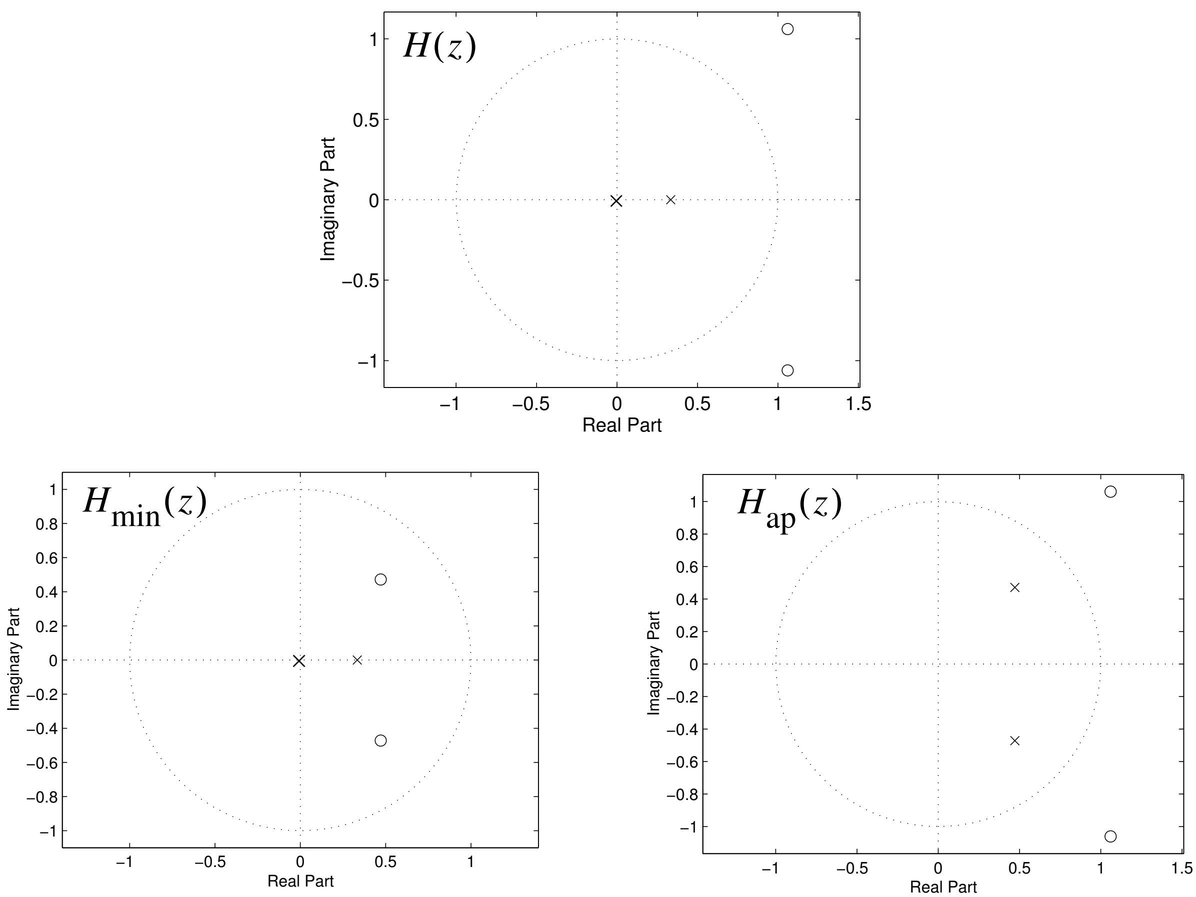
- To reflect the conjugate zeros at $\pm\angle\pi/4$ we first factor the zeros out of the numerator to prepare for a second-order all-pass section

$$H(z) = \left(\frac{3}{2}\right)^2 \frac{\left(z^{-1} - \frac{2}{3}e^{-j\pi/4}\right)\left(z^{-1} - \frac{2}{3}e^{j\pi/4}\right)}{1 - \frac{1}{3}z^{-1}}$$

Note that $e^{-j\pi/4} \cdot e^{j\pi/4} = 1$

- Next we introduce the all-pass section following the minimum phase equivalent

$$
\begin{aligned}
H(z) &= H_{\text{min}}(z) H_{\text{ap}}(z) \\
H(z) &= \left[\frac{9}{4} \frac{\left(1 - \frac{2}{3}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{2}{3}e^{-j\pi/4}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}} \right] \\
&\quad \times \left[\frac{\left(z^{-1} - \frac{2}{3}e^{-j\pi/4}\right)\left(z^{-1} - \frac{2}{3}e^{j\pi/4}\right)}{\left(1 - \frac{2}{3}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{2}{3}e^{-j\pi/4}z^{-1}\right)} \right]
\end{aligned}
$$



Pole-zero plots of $H(z)$, $H_{\text{min}}(z)$, and $H_{\text{ap}}(z)$

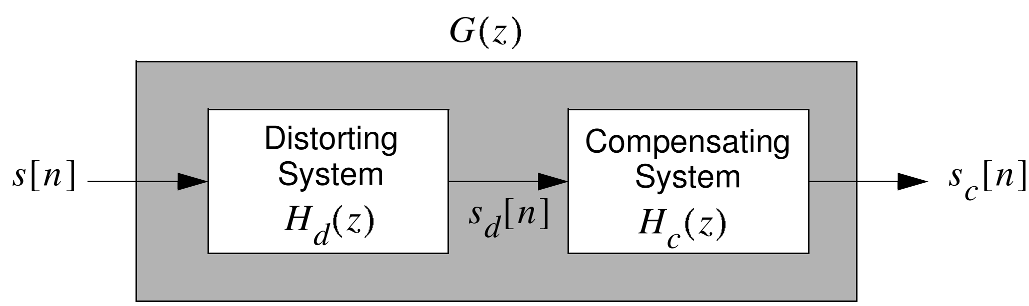
5.6.1 Frequency Response Compensation

A signal processing application where a stable and causal inverse is required is in communications channel equalization. Suppose transmitted signal $s[n]$ is observed at the receiver as $s_d[n]$ after passing through a stable and causal system $H_d(z)$. We wish to filter $s_d[n]$ to recover as near as possible the original $s[n]$.

- If $H_d(z)$ is minimum-phase then a stable and causal inverse will exist and $s[n]$ can be recovered

- If H_d is nonminimum-phase then we can phase compensate it with an allpass filter $H_{\text{ap}}(z)$ to form a minimum-phase system $H_{\min}(z) = H_d(z)/\,H_{\text{ap}}(z)$

- To compensate the channel amplitude response we can now use the stable and causal filter $H_c(z) = 1/H_{\min}(z)$



Compensation system

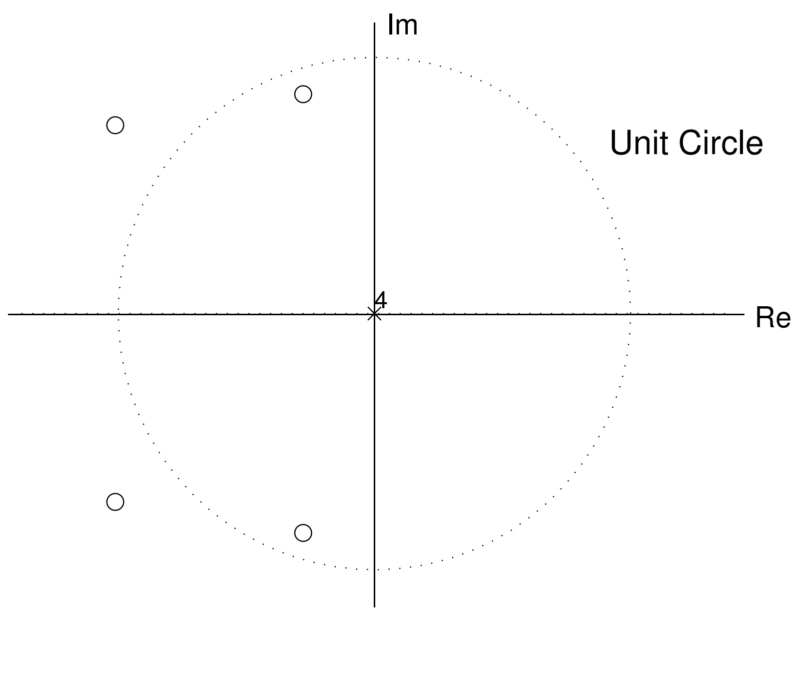
- The total system function relating the input $s[n]$ to the compensated output $s_c[n]$ is

$$G(z) = H_d(z)H_c(z) = H_{\text{ap}}(z)$$

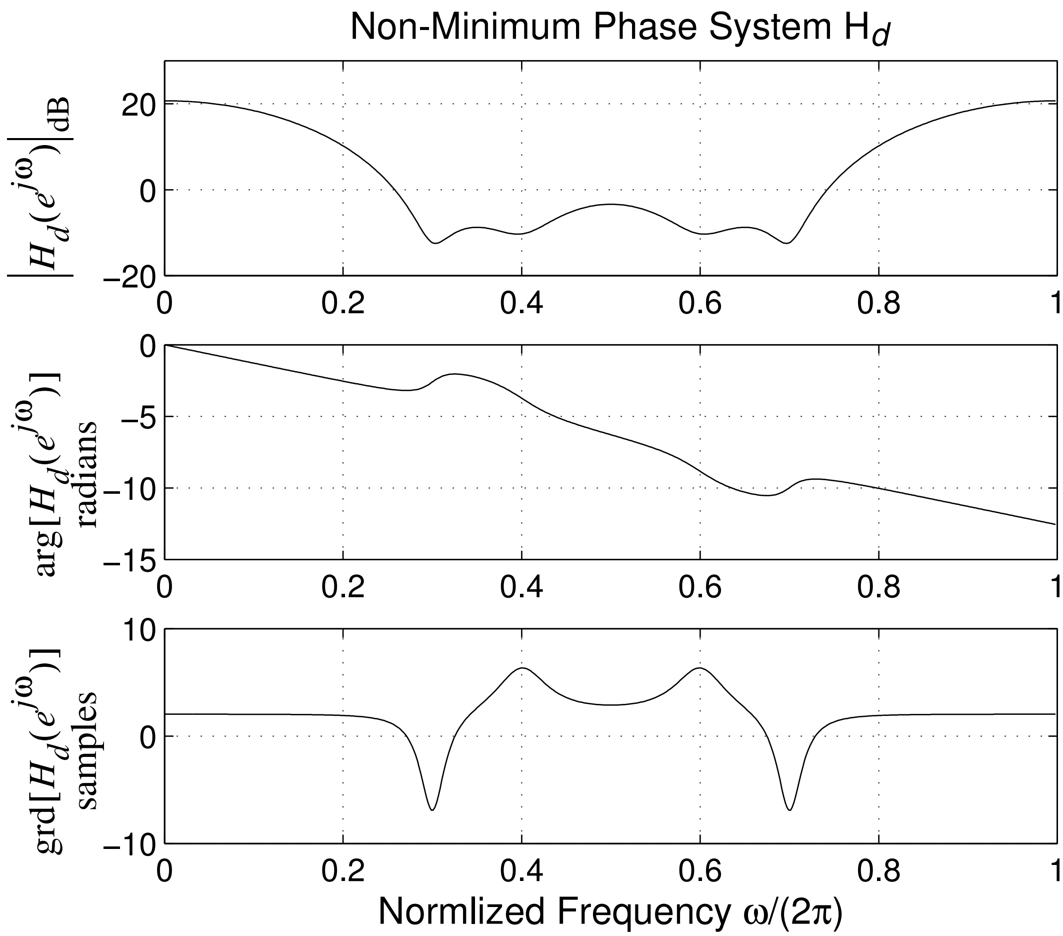
Example 5.11: Compensation of an FIR System

Consider the following FIR system

$$
\begin{aligned}
H_d(z) &= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1}) \\
&\quad \times (1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1}) \\
&= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1}) \\
&\quad \times (z^{-1}-0.8e^{-j0.8\pi})(z^{-1}-0.8e^{j0.8\pi})(1.25)^2
\end{aligned}
$$



FIR system $H_d(z)$ pole-zero plot

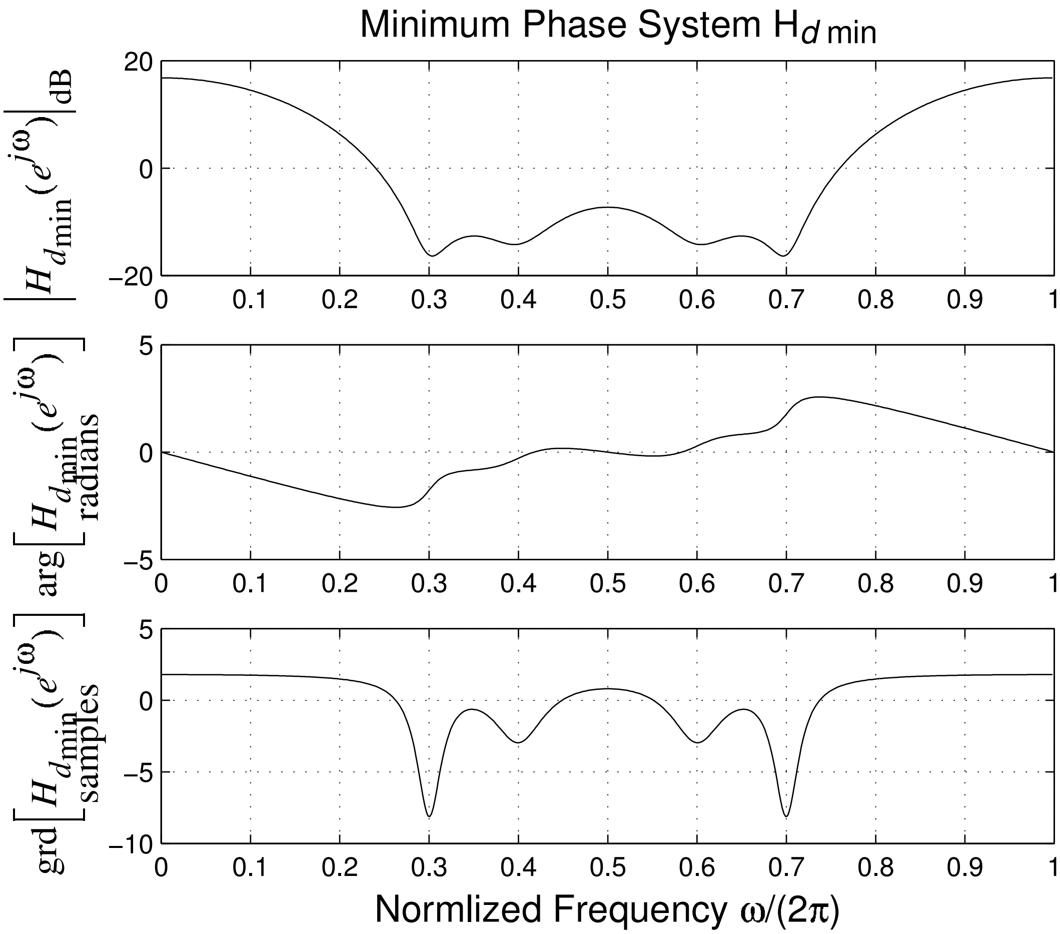


Magnitude, phase, and group delay of $H_d(e^{j\omega})$

- From the pole-zero plot we see that the zeros at $z = 1.25e^{\pm j0.8\pi}$ make the system nonminimum-phase

- To compensate $H_d(z)$ we first must find the corresponding minimum phase system, $H_{d\ \mathrm{min}}(z)$, which is obtained by reflecting the zeros at $z = 1.25e^{\pm j0.8\pi}$ to their conjugate reciprocal locations inside the unit circle

$$
\begin{aligned}
&= (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1}) \\
&\quad \times (1 - 0.8e^{j0.8\pi}z^{-1})(1 - 0.8e^{-j0.8\pi}z^{-1})(1.25)^2
\end{aligned}
$$



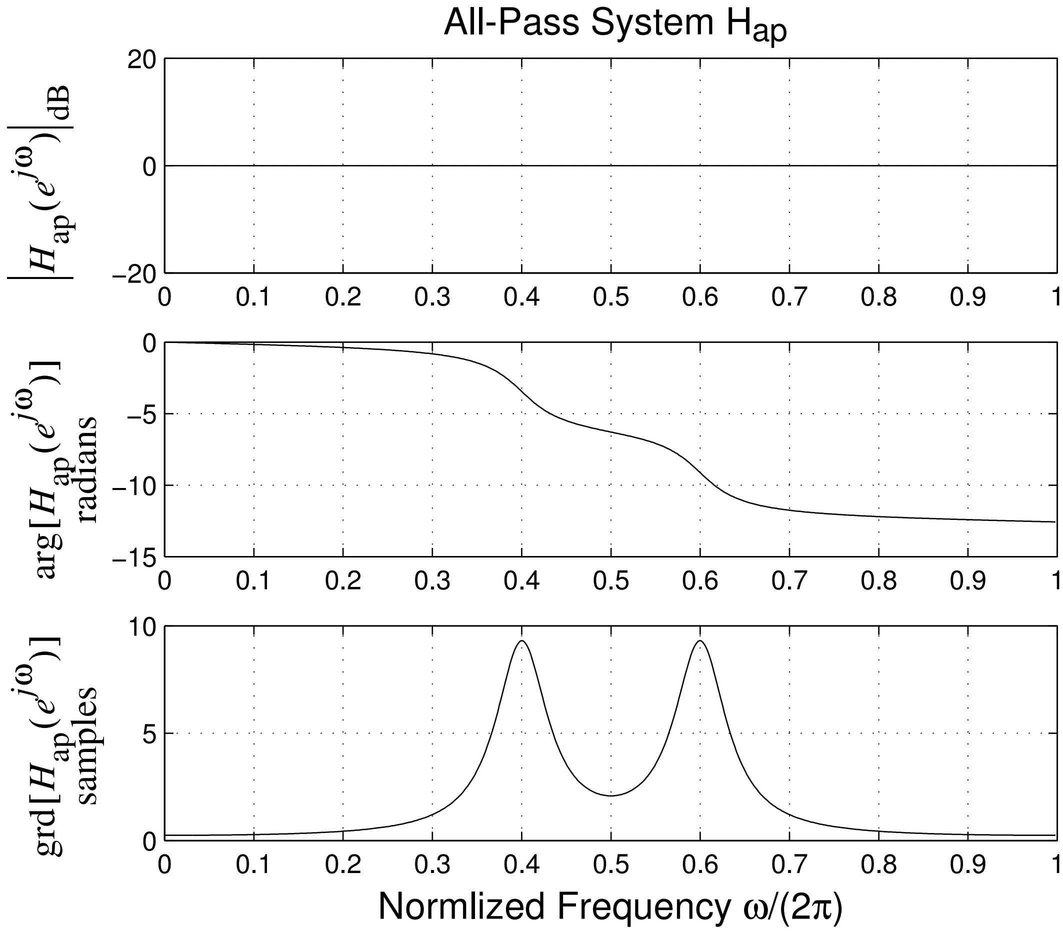
Magnitude, phase, and group delay of $H_{d\ \mathrm{min}}(e^{j\omega})$

- The all-pass system that relates $H_{d\ \mathrm{min}}(z)$ to $H_d(z)$ is

$$H_{\mathrm{ap}}(z) = \frac{\left(z^{-1} - 0.8e^{-j0.8\pi}\right)\left(z^{-1} - 0.8e^{j0.8\pi}\right)}{\left(1 - 0.8e^{j0.8\pi}z^{-1}\right)\left(1 - 0.8e^{-j0.8\pi}z^{-1}\right)}$$

that is $H_d(z) = H_{\mathrm{ap}}(z)H_{d\ \mathrm{min}}(z)$

- Since the causal inverse of $H_d(z)$ is unstable, the minimum phase inverse would have to be used for $H_c(z)$, and the net system function that results is $G(z) = H_{\mathrm{ap}}(z)$



Magnitude, phase, and group delay of $H_{\text{ap}}(e^{j\omega}) \overset{\text{also}}{=} G(z)$

5.6.2 Properties of Minimum-Phase Systems

In this section several properties of minimum-phase systems will be discussed.

Minimum Phase-lag Property:

For any system $H(z)$ we can write the continuous phase as

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{\text{ap}}(e^{j\omega})],$$

which is the sum of the continuous phase of a minimum-phase system and the continuous phase of an allpass system.

- We know that the continuous phase of an allpass system is negative for $0 < \omega \leq \pi$ hence $\arg[H(e^{j\omega})]$ must always be more negative than $\arg[H_{\min}(e^{j\omega})]$
- Stated another way the *phase-lag* of a nonminimum-phase system $H(z)$ is always greater than the corresponding minimum-phase system $H_{\min}(z)$
- A more precise term for a minimum-phase system is a *minimum phase-lag* system

Minimum Group Delay Property:

The group delay for any system can be written as

$$\text{grd}[H(e^{j\omega})] = \text{grd}[H_{\min}(e^{j\omega})] + \text{grd}[H_{\text{ap}}(e^{j\omega})]$$

- For a rational allpass system the group delay is always positive, thus for any system with magnitude response $|H_{\min}(e^{j\omega})|$, the group delay of the corresponding minimum-phase system has the minimum group delay
- A minimum-phase system can also be referred to as a *minimum group delay* system

Minimum Energy Delay Property:

- It can be shown that for any causal and stable system for which

$$|H(e^{j\omega})| = |H_{\min}(e^{j\omega})|$$

then the corresponding impulse responses satisfy

$$|h[0]| \le |h_{\min}[0]|$$

- Define the partial energy of a causal impulse response as

$$E[n] = \sum_{m=0}^{n} |h[m]|^2$$

- Note that in the limit as $n \to \infty$ the partial energy becomes the total energy

- Since $|H(e^{j\omega})|^2 = |H_{\min}(e^{j\omega})|^2$, from Parseval's theorem it follows that both impulse responses have the same total energy

- The *minimum energy delay property* states that

$$E[n] = \sum_{m=0}^{n} |h[m]|^2 \le \sum_{m=0}^{n} |h_{\min}[m]|^2 = E_{\min}[n]$$

  for all $h[n]$'s belonging to the family of systems having magnitude responses equal to $|H_{\min}(e^{j\omega})|$. A proof for this is given in O.S. problem 5.65.

- To prove that $|h[0]| \le |h_{\min}[0]|$ consider the special case where $H(z)$ and $H_{\min}(z)$ are related by a first-order allpass system with real pole a

$$H(z) = H_{\min}(z)\frac{z^{-1} - a}{1 - az^{-1}}, \quad |a| < 1$$

- Using the initial value theorem we can write

$$\lim_{z\to\infty} H_{\min}(z) = \lim_{z\to\infty} \frac{1 - az^{-1}}{z^{-1} - a} H(z)$$

or after evaluating both limits

$$h_{\min}[0] = \frac{1}{-a} h[0] \Rightarrow |h_{\min}[0]| > |h[0]|$$

since $|a| < 1$

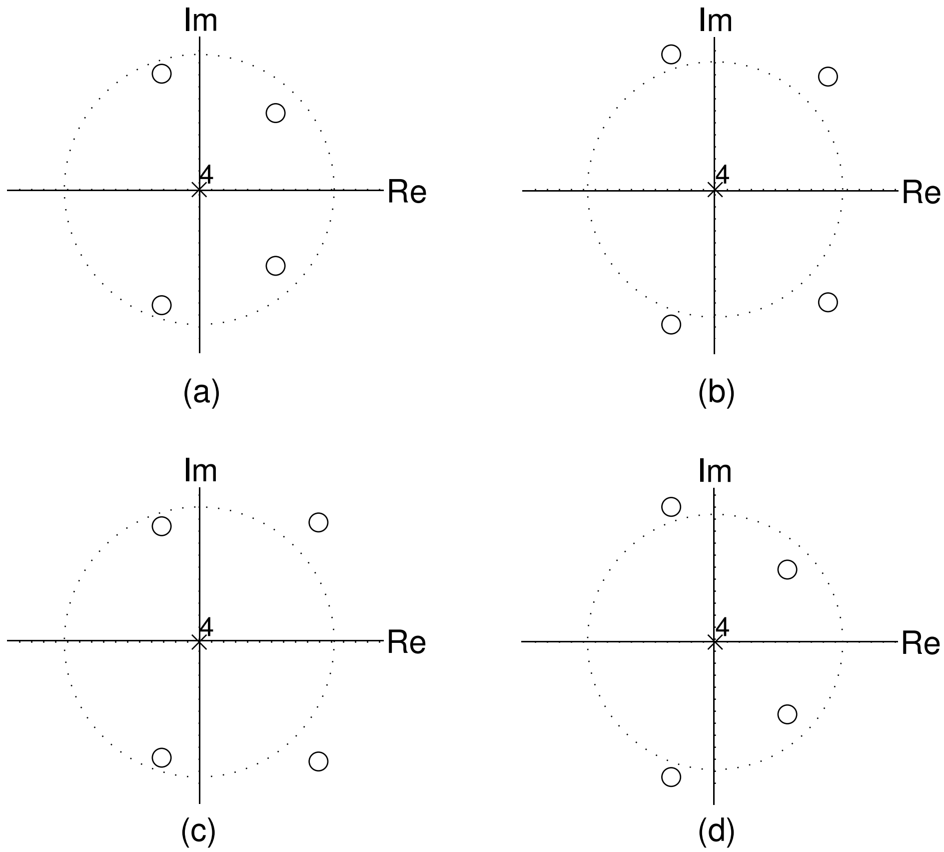
- This process can be repeated for each additional first-order all-pass section needed to transform $H(z)$ to $H_{\min}(z)$

- Note that the minimum energy delay property states that the energy of a minimum-phase system is delayed least of all systems having the same magnitude response, hence they are also termed *minimum energy delay* systems

Example 5.12: Fourth-order FIR System

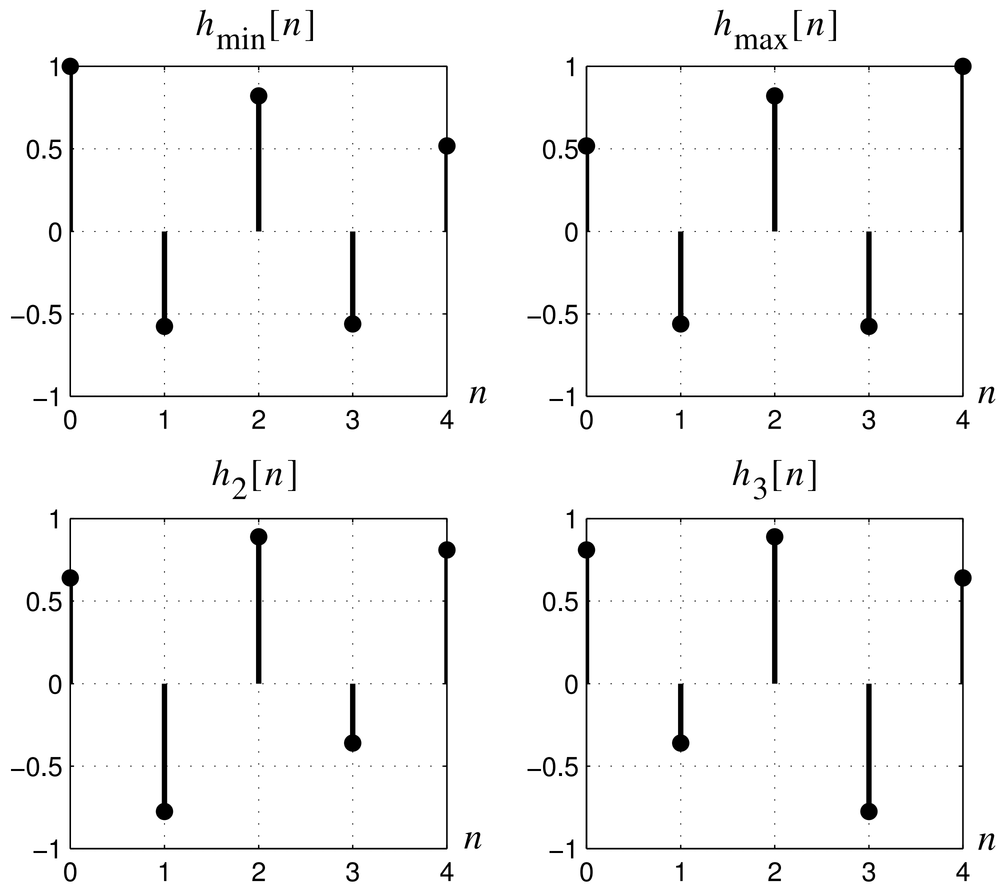
Consider the four possible pole-zero plots associated with the fourth order minimum phase FIR system

$$\begin{aligned} H_{\min}(z) = &\big(1 - 0.9e^{j0.6\pi}z^{-1}\big)\big(1 - 0.9e^{-j0.6\pi}z^{-1}\big) \\ &\times \big(1 - 0.8e^{j0.25\pi}z^{-1}\big)\big(1 - 0.8e^{-j0.25\pi}z^{-1}\big) \end{aligned}$$



Four-pole FIR system pole-zero plots (a) minimum phase, (b) maximum phase, (c&d) mixed phase

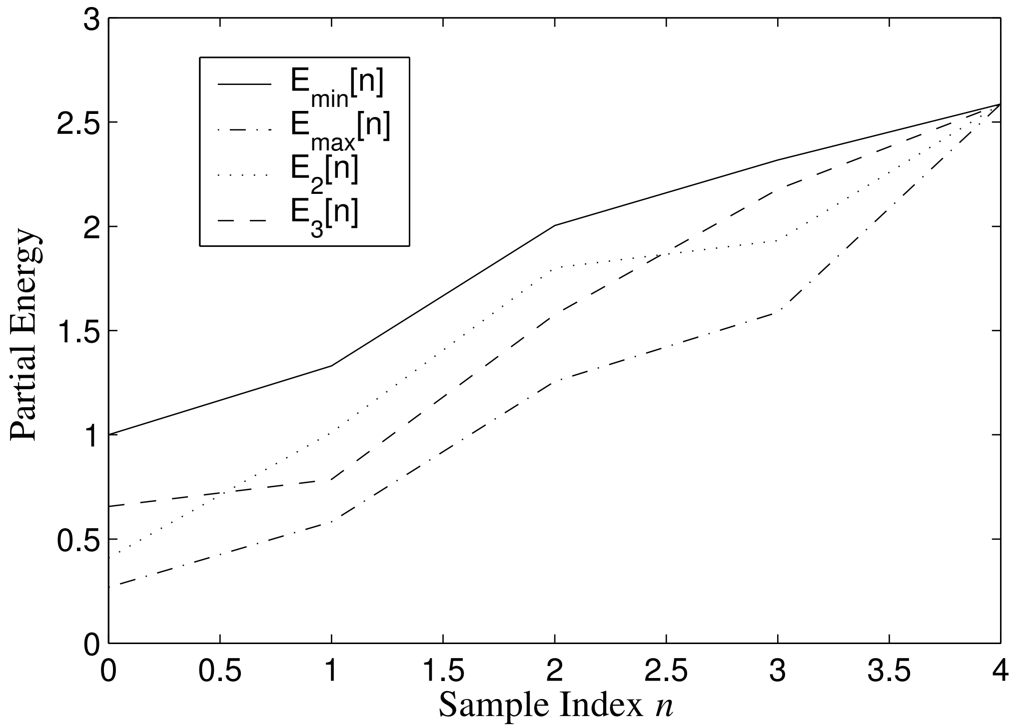
- The corresponding impulse responses are given below



Four-pole FIR system impulse responses (a) minimum phase, (b) maximum phase, (c&d) mixed phase

- The partial energies of the four impulse responses are plotted below

- Note that the minimum and maximum-phase systems have partial energy functions which bound the remaining partial energies



Four-pole FIR system partial energy sequences

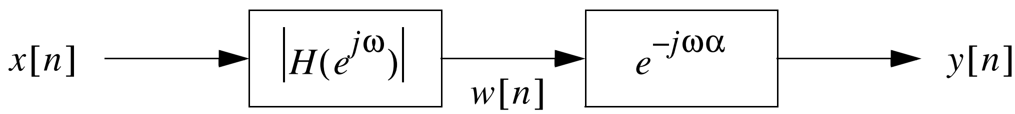
5.7 Linear Systems with Generalized Linear Phase

In signal processing we often wish to pass some portion of a desired frequency band with little or no distortion. With a causal system constraint this implies a flat amplitude passband with linear phase is most desirable.

5.7.1 Ideal Linear Phase Systems

An ideal linear phase system can be viewed as a cascade system consisting of a nonconstant magnitude response combined with a linear phase term corresponding to delay α (in general α is real)

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}$$



Linear Phase System with Arbitrary Magnitude Response

Example 5.13: **Ideal Lowpass Filter**

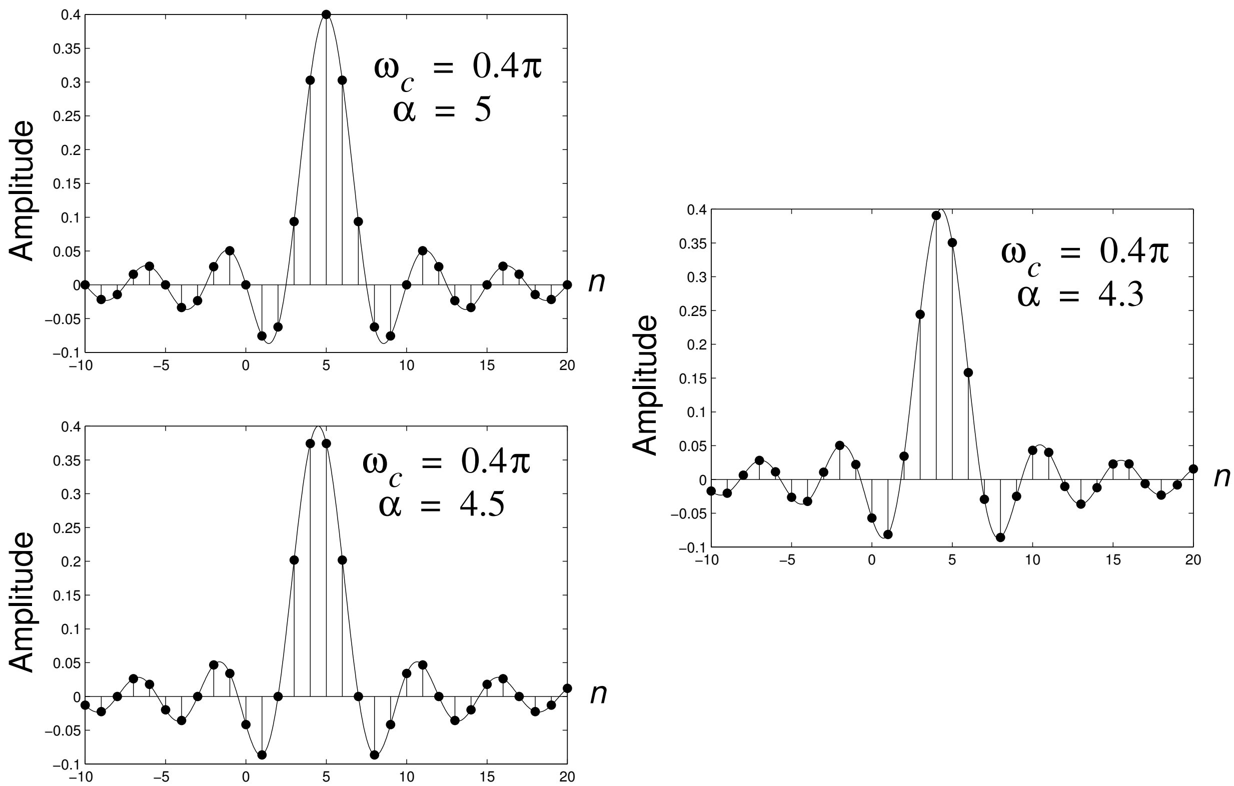
An ideal lowpass filter with linear phase can be written as

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

- The impulse response can be written as

$$h_{lp}[n] = \frac{\sin[\omega_c(n-\alpha)]}{\pi(n-\alpha)}$$

- The following figure illustrates how the linear phase corresponding to both integer and noninteger delays is manifested in the impulse response



5.7.2 Generalized Linear Phase

Recall that a causal moving average system has frequency response

$$H(e^{j\omega}) = \frac{1}{M+1}\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}e^{-j\omega M/2}, \quad |\omega| < \pi$$

- Note that the phase of this system is not strictly linear since the $\sin[\omega(M+1)/2]/\sin(\omega/2)$ term contributes $\pm\pi$ radians when its sign becomes negative

- Systems with this sort of deviation from linear phase still possess many of the advantages of true linear phase systems

Generalized Linear Phase: A system with frequency response of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta},$$

α and β constants and $A(e^{j\omega})$ real (possibly bipolar), is said to be a generalized linear phase system.

- The group delay of this system is

$$\text{grd}[H(e^{j\omega})] = \alpha$$

  assuming we ignore the discontinuities resulting from a constant phase over all or part of the band $0 < \omega < \pi$

- Using the frequency response of the generalized linear phase system we can derive corresponding conditions that $h[n]$ must satisfy

- To establish the conditions on $h[n]$ start by writing

$$\begin{aligned} H(e^{j\omega}) = {} & A(e^{j\omega})\cos(\beta - \omega\alpha) \\ & + jA(e^{j\omega})\sin(\beta - \omega\alpha) \end{aligned}$$

- Assuming $h[n]$ is real we can then also write

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]\cos\omega n - j\sum_{n=-\infty}^{\infty} h[n]\sin\omega n \end{aligned}$$

- Equating the two forms of $H(e^{j\omega})$ as written above we see that the tangent of $\angle H(e^{j\omega})$ must satisfy

$$\begin{aligned} \tan(\beta - \omega\alpha) &= \frac{\sin(\beta - \omega\alpha)}{\cos(\beta - \omega\alpha)} \\ &= \frac{-\sum_{n=-\infty}^{\infty} h[n]\sin\omega n}{\sum_{n=-\infty}^{\infty} h[n]\cos\omega n} \end{aligned}$$

or

$$\sum_{n=-\infty}^{\infty} h[n] \sin[\omega(n-\alpha)+\beta] = 0, \quad \forall\, \omega$$

- The above is a necessary condition, but is not a sufficient condition for insuring linear phase. Note this is not really a design equation!

5.7.3 Causal Generalized Linear-Phase FIR Systems

- If we further require that the system be causal, then the necessary condition becomes

$$\sum_{n=0}^{\infty} h[n] \sin[\omega(n-\alpha)+\beta] = 0 \quad \forall\, \omega$$

- The simplest systems which satisfy the generalized linear phase and causal condition are FIR of length $M+1$

- One possibility (assuming real coefficients) is *even symmetric* coefficients

$$h[n] = \begin{cases} h[M-n], & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

where here $\beta = 0$ or π

- The frequency response will be of the form

$$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where $A_e(e^{j\omega})$ is a real even function of ω

- A second possibility is *odd symmetric* coefficients

$$h[n] = \begin{cases} -h[M-n], & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

where here $\beta = \pi/2$ or $3\pi/2$

- The frequency response will be of the form

$$H(e^{j\omega}) = jA_o(e^{j\omega})e^{-j\omega M/2} = A_o(e^{j\omega})e^{-j\omega M/2 + j\pi/2}$$

where $A_o(e^{j\omega})$ is a real and odd function of ω

- The even and odd symmetry length $M+1$ FIR systems can each be specialized into two distinct types depending upon M being an even or odd integer; four total types

Type I FIR Linear Phase Systems:

Here $h[n]$ is symmetric i.e.,

$$\boxed{h[n] = h[M-n], \;\; 0 \leq n \leq M}$$

and M is assumed even.

- To obtain the frequency response we begin by writing

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= e^{-j\omega M/2}\Big\{h[M/2] + \sum_{n=0}^{\frac{M}{2}-1}\big[h[n]e^{-j\omega(n-M/2)} \\
&\quad + h[M-n]e^{-j\omega(M/2-n)}\big]\Big\} \\
&= e^{-j\omega M/2}\Big\{h[M/2] + \sum_{k=1}^{M/2}\big[h[(M/2)-k]e^{j\omega k} \\
&\quad + h[(M/2)+k]e^{-j\omega k}\big]\Big\}
\end{aligned}
$$

- Finally,

$$
\begin{aligned}
H(e^{j\omega}) = e^{-j\omega M/2}\Big\{&h[M/2] \\
&+ 2\sum_{k=1}^{M/2} h[(M/2)-k]\cos\omega k\Big\}
\end{aligned}
$$

  where the substitution $k = (M/2) - n$ has been used

- If we define a new set of coefficients $a[k],\ k = 0,\ 1, \ldots, M/2$ as

$$
\begin{aligned}
a[0] &= h[M/2] \\
a[k] &= 2h[(M/2)-k],\quad k = 1, 2, \ldots, M/2
\end{aligned}
$$

then we can write

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=0}^{M/2} a[k]\cos\omega k\right)$$

- Note that $H(e^{j\omega})$ is of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta},\ \alpha = M/2,\ \beta = 0 \text{ or } \pi$$

Type II FIR Linear Phase Systems:

Here $h[n]$ is again even, but now M is assumed odd.

- It can be shown that

$$H(e^{j\omega}) = e^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} b[k]\cos[\omega(k-1/2)]\right)$$

where the $b[k]$ coefficients are

$$b[k] = 2h[(M+1)/2 - k],\ \ k = 1, 2, \ldots, (M+1)/2$$

- Note that $H(e^{j\omega})$ is of the same form as the Type I system except now α is an integer plus one-half

Type III FIR Linear Phase Systems:

Here $h[n]$ is odd symmetric or antisymmetric

$$h[n] = -h[M-n],\ \ 0 \le n \le M$$

and M is assumed even.

- It can be shown that

$$H(e^{j\omega}) = je^{-j\omega M/2}\left(\sum_{k=1}^{M/2} c[k]\sin\omega k\right)$$

  where the $c[k]$ coefficients are

$$c[k] = 2h[(M/2) - k], \;\; k = 1, 2, \ldots, M/2$$

- Note that $H(e^{j\omega})$ is of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}, \;\; \beta = \pi/2 \text{ or } 3\pi/2$$

- Systems with antisymmetric impulse response are useful in the design of wideband differentiators and Hilbert transformers

Type IV FIR Linear Phase Systems:

Here $h[n]$ is again antisymmetric, but now M is assumed odd.

- It can be shown that

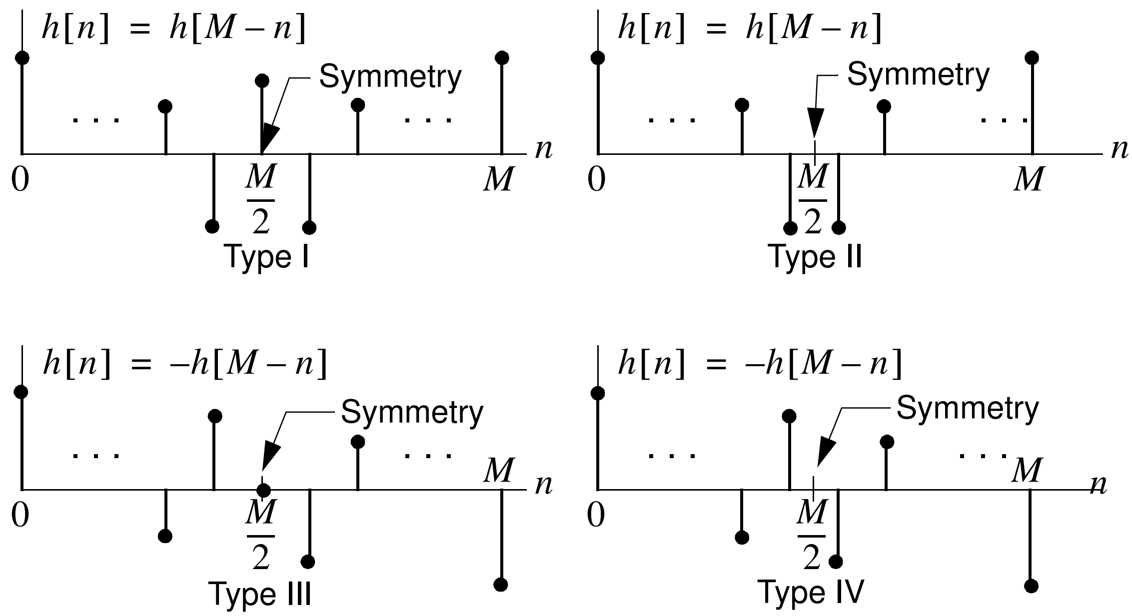
$$H(e^{j\omega}) = je^{-j\omega M/2}\left(\sum_{k=1}^{(M+1)/2} d[k]\sin[\omega(k - 1/2)]\right)$$

  where the $d[k]$ coefficients are

$$d[k] = 2h[(M+1)/2 - k], \;\; k = 1, 2, \ldots, (M+1)/2$$

- Note that $H(e^{j\omega})$ is of the same form as the Type III system except now α is an integer plus one-half

Summary of Symmetry Conditions



Generic display of type I–IV symmetry conditions on $h[n]$

Zero Locations for FIR Type I and II Systems

- For $h[n]$ real symmetric we can write

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} = \sum_{n=0}^{M} h[M-n]z^{-n}$$

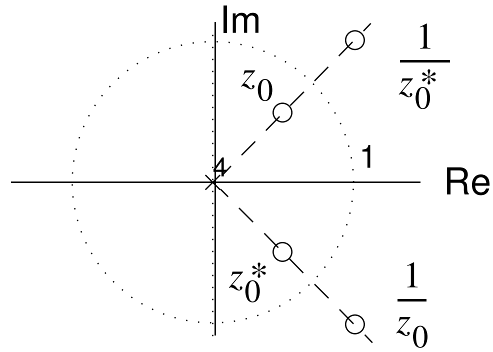
$$= \sum_{k=M}^{0} h[k]z^{k-M} = z^{-M}H(z^{-1})$$

- Let $z_o = r_o e^{j\theta_o}$ be one of the M zeros of $H(z)$
- Since $H(z) = z^{-M}H(z^{-1})$, in particular

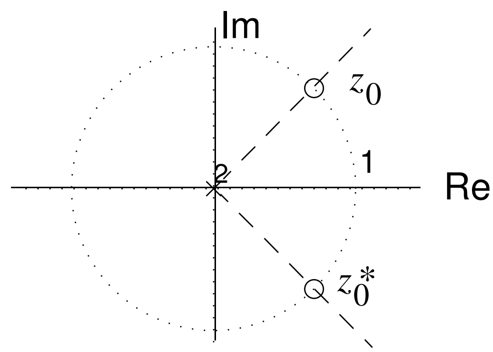
$$H(z_o) = z_o^{-M}H(z_o^{-1}) \stackrel{\text{also}}{=} 0$$

so if z_o is a zero then so is $z_o^{-1} = r_o^{-1}e^{-j\theta_o} \Longrightarrow$

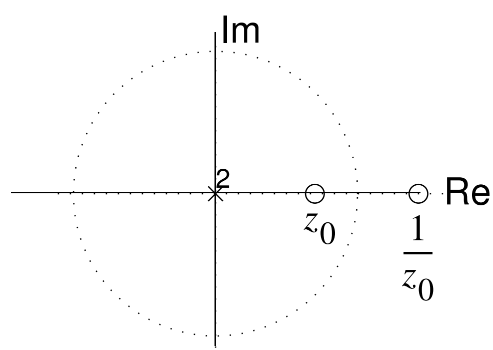
– If $r_o \neq 1$ and $\theta_o \neq 0$ or π, then the zeros occur in quadruplets



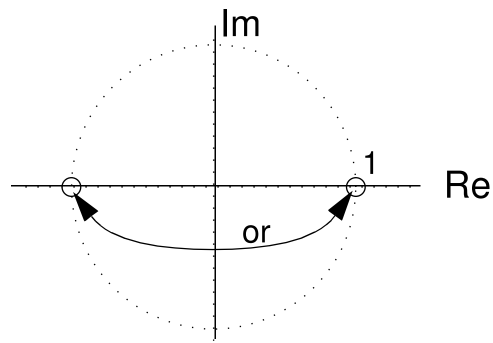
– If $r_o = 1$ and $\theta_o \neq 0$ or π, then the zeros occur in conjugate pairs on the unit circle



– If $r_o \neq 1$ and $\theta_o = 0$ or π, then the zeros are real and occur in reciprocal pairs



– Note that if $r_o = 1$ and $\theta_o = 0$ or π, then the zero may appear by itself at either ± 1



- For the case case M = odd (Type II) we must have a zero at $z = -1$ to satisfy

$$H(-1) = (-1)^{-M} H(-1) = -H(-1)$$

$$\Rightarrow \; H(-1) = 0$$

- Note that a highpass filter would not be realized with M odd since $H(e^{j\pi})$ is constrained to be zero

Zero Locations for FIR Type III and IV Systems

- For $h[n]$ antisymmetric we can write

$$H(z) = -z^{-M} H(z^{-1})$$

- The zero locations are constrained as in the symmetric case with the additional constraint that $H(z)$ must always have a zero at $z = 1$. Why?

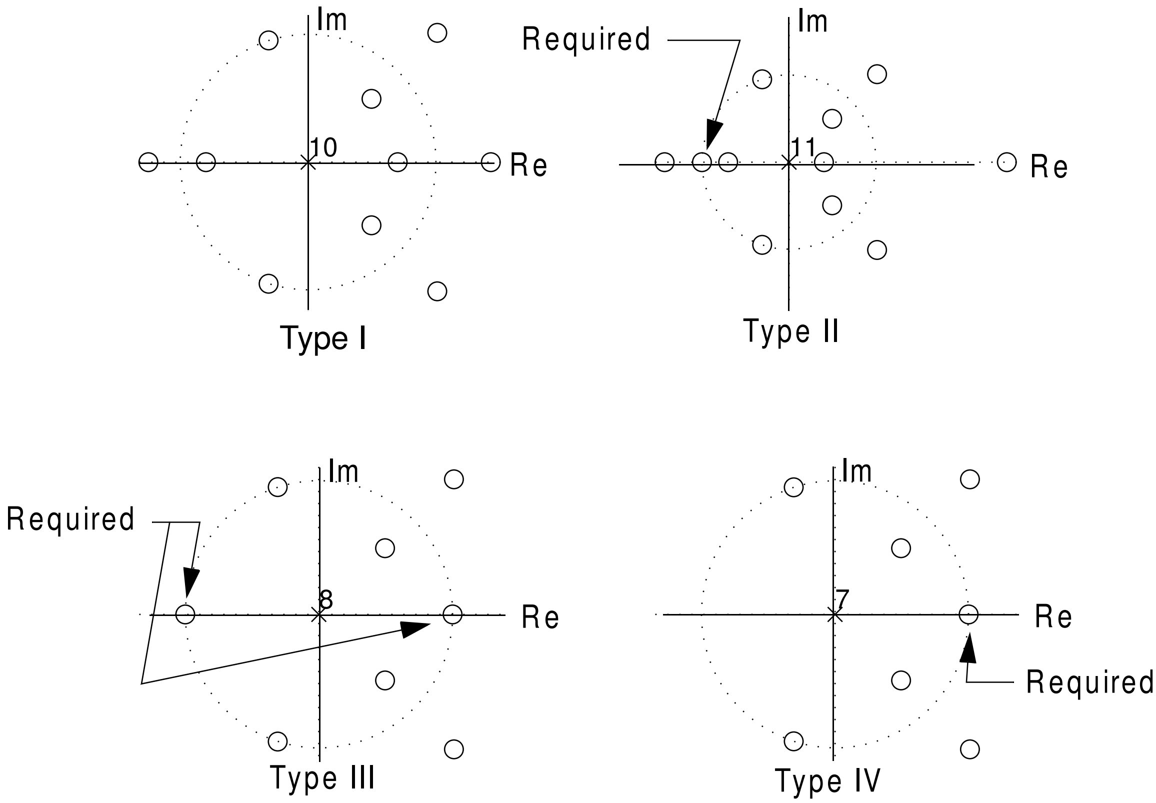
$$H(1) = -(1)^{-M} H(1) = -H(1) \quad \Rightarrow \; H(1) = 0$$

- If M is also even, then we must also have a zero at $z = -1$

$$H(-1) = -(-1)^{-M} H(-1) = -H(-1)$$

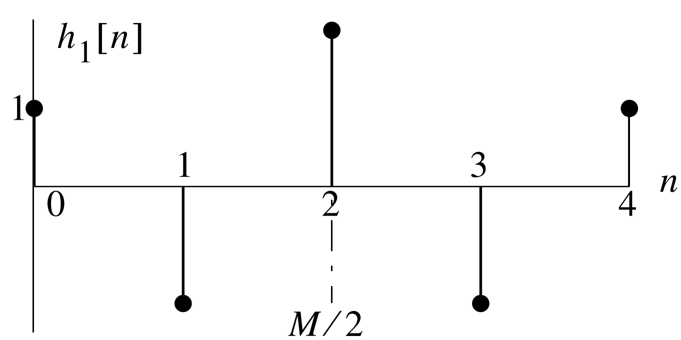
$$\Rightarrow \; H(-1) = 0$$

Summary of Linear Phase FIR Zero Plots



(a) Type I (b) Type II (c) Type III (d) Type IV

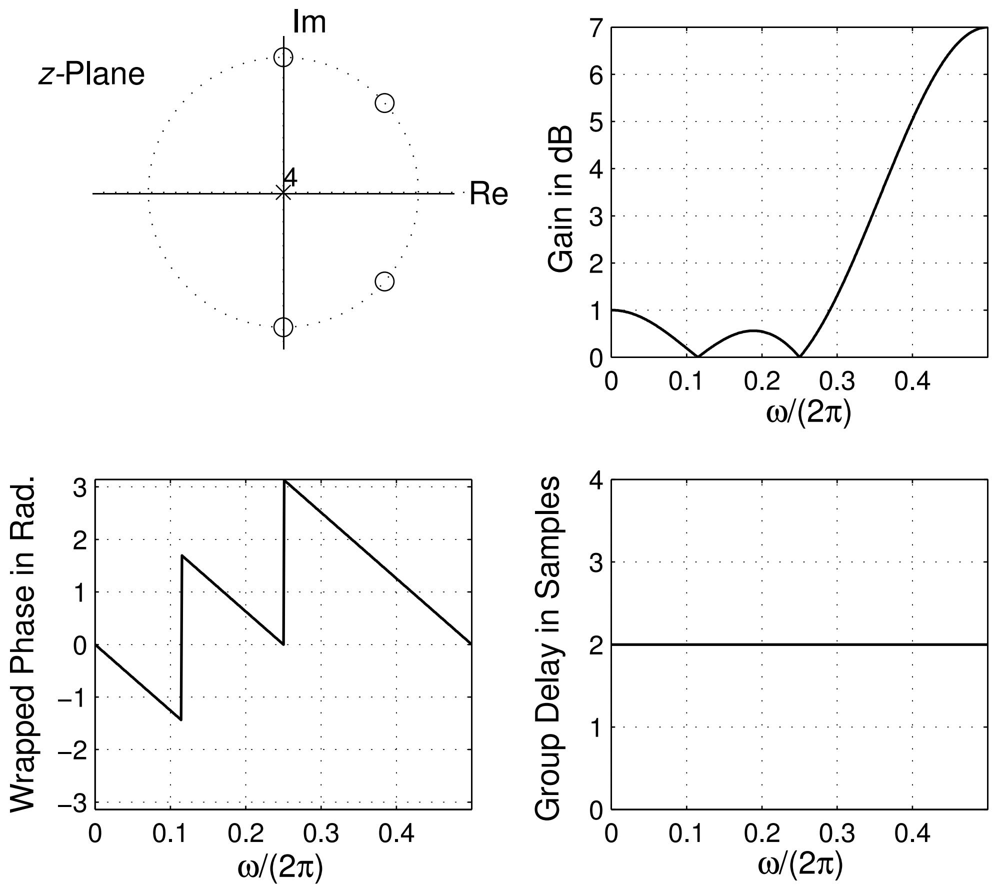
Example 5.14: Type I with $M = 4$



A Type I impulse response with $M = 4$

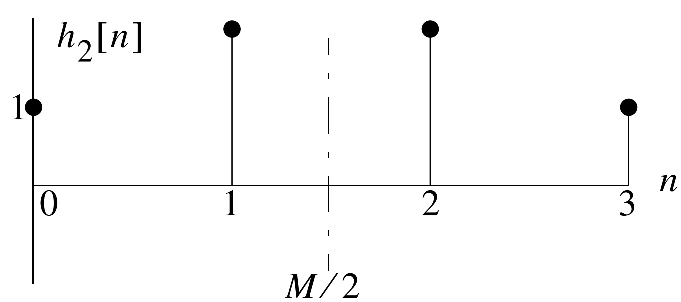
- The corresponding system function is

$$H_1(z) = 1 - 1.5z^{-1} + 2z^{-2} - 1.5z^{-3} + z^{-4}$$



Pole-zero, magnitude, phase, and group delay for Type I
$M = 4$

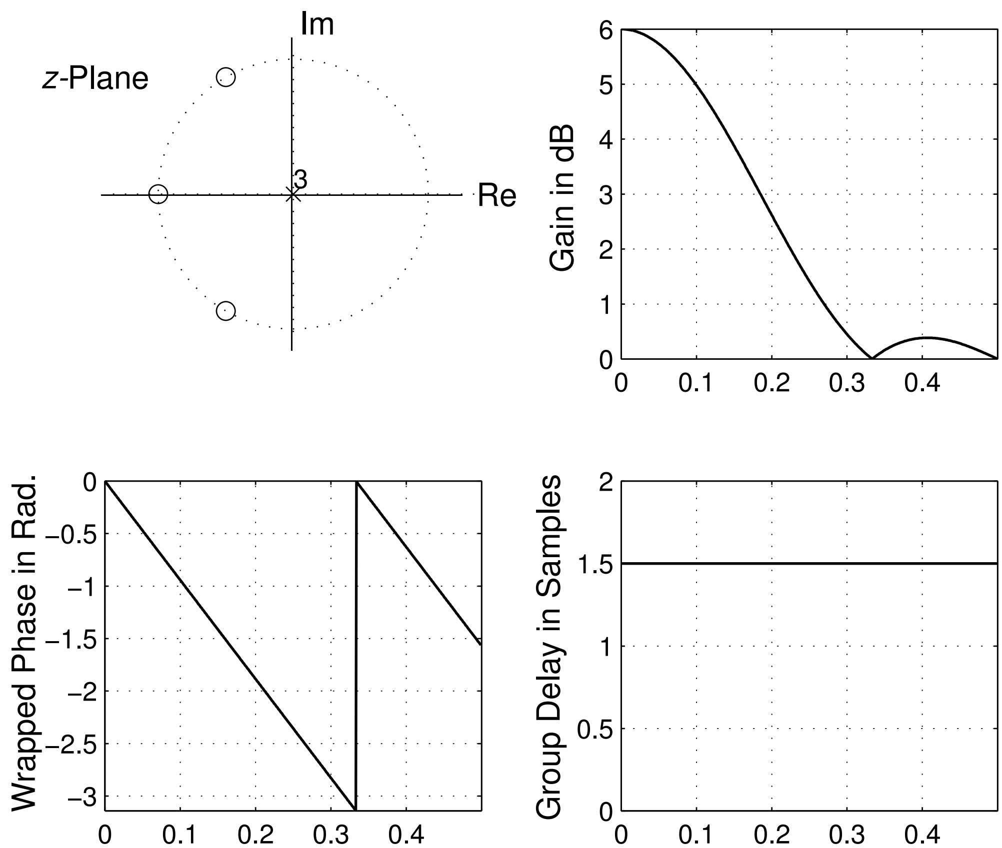
Example 5.15: **Type II with $M = 3$**



A Type I impulse response with $M = 3$

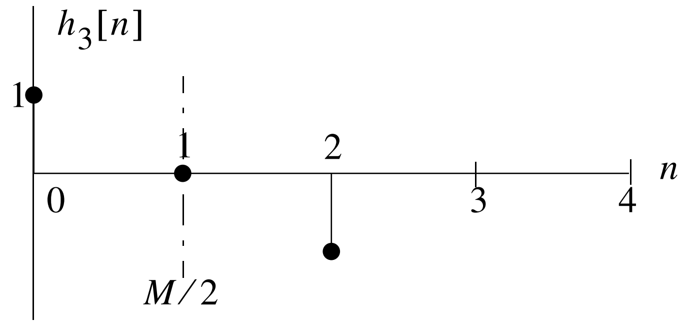
- The corresponding system function is

$$H_2(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$



Pole-zero, magnitude, phase, and group delay for Type II $M = 3$

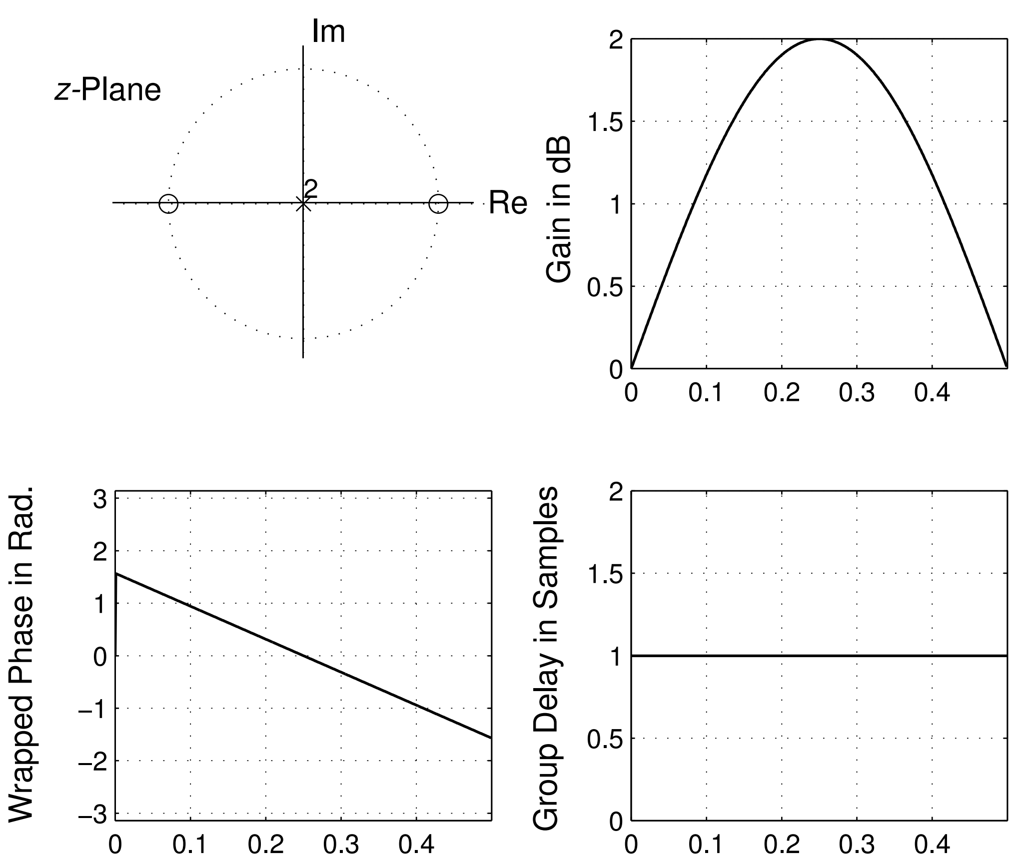
Example 5.16: **Type III with $M = 2$**



A Type II impulse response with $M = 2$

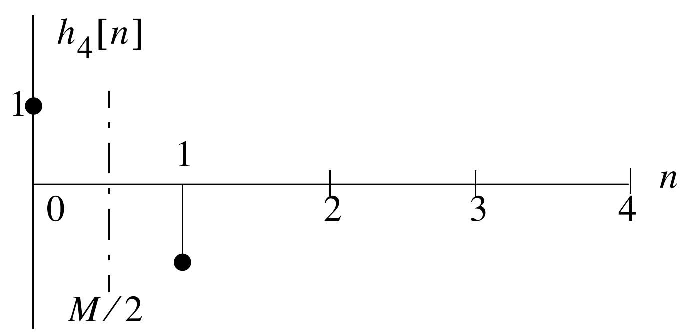
- The corresponding system function is

$$H_3(z) = 1 - z^{-2}$$



Pole-zero, magnitude, phase, and group delay for Type III
$M = 2$

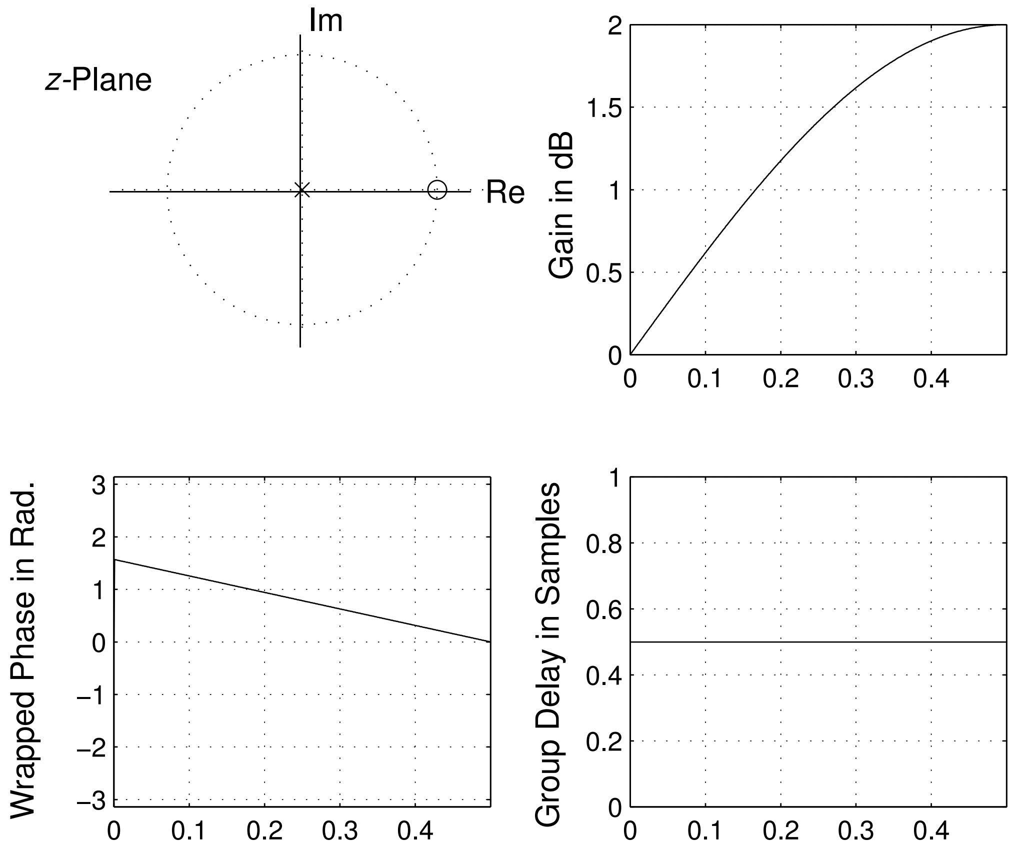
Example 5.17: **Type IV with $M = 1$**



A Type IV impulse response with $M = 1$

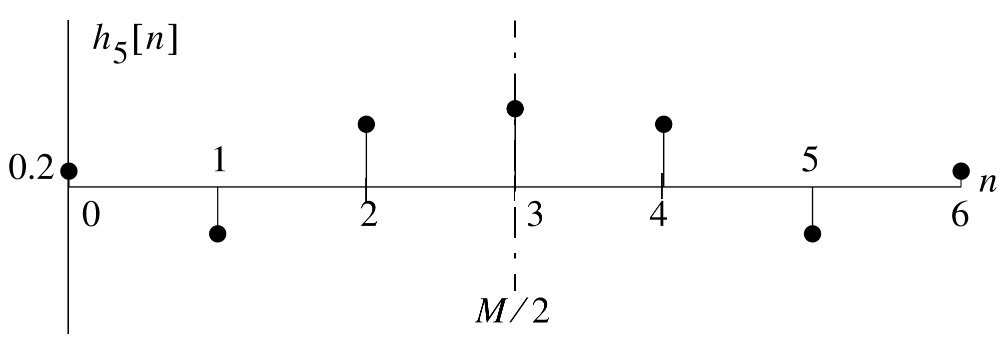
- The corresponding system function is

$$H_4(z) = 1 - z^{-1}$$



Pole-zero, magnitude, phase, and group delay for Type IV
$M = 1$

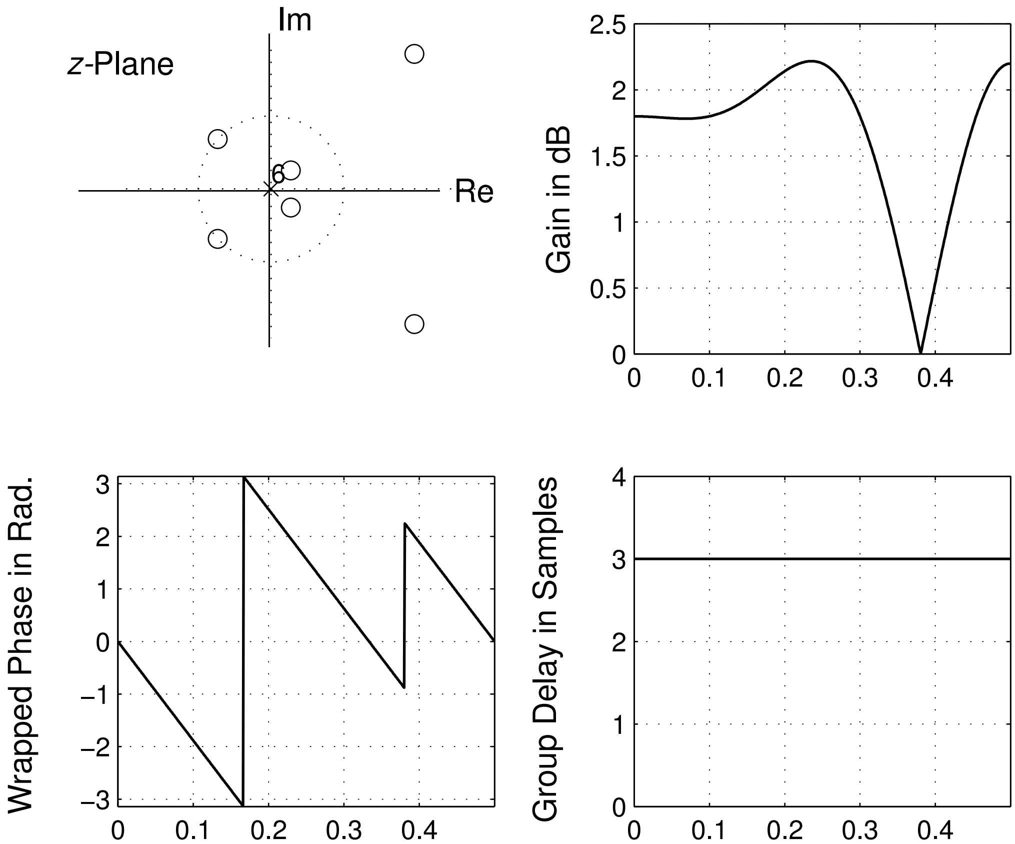
Example 5.18: **Type I with $M = 6$**



A Type I impulse response with $M = 6$

- The corresponding system function is

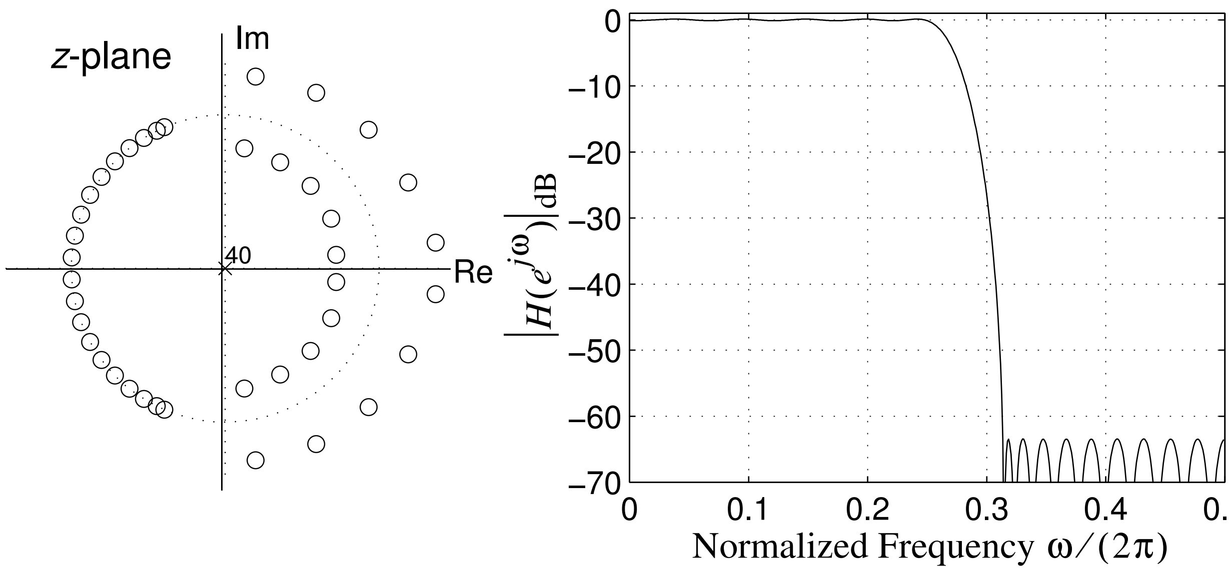
$$H_5(z) = 0.2 - 0.6z^{-1} + 0.8z^{-2} + z^{-3} + 0.8z^{-4}$$
$$- 0.6z^{-5} + 0.2z^{-6}$$



Pole-zero, magnitude, phase, and group delay for Type I $M = 6$

Example 5.19: A Type I $M = 40$ equiripple lowpass filter

Using the MATLAB function `remez()` we obtain the following design:



(a) Magnitude response (b) pole-zero plot

Example 5.20: Frequency Sampling FIR Design

Suppose we wish to design a linear phase FIR filter of Type I using a $\frac{M}{2} + 1$ frequency sampling (matching) specification.

- For a Type I system we are free to specify $A(e^{j\omega})$ at $\frac{M}{2} + 1$ points spaced over the interval $[0, \pi]$
- Note if M were odd, then we would be forced to choose $A(e^{j\pi}) = 0$, thus eliminating one degree of freedom in specifying the frequency response

- To obtain the $a[k]$'s we have a system of $\frac{M}{2}+1$ equations to solve i.e.

$$\sum_{k=0}^{M/2} a[k]\cos\omega_n k = A(e^{j\omega_n}),\ n = 0,1,\ldots,M/2$$

or in matrix form we must solve

$$\mathbf{Ca} = \mathbf{A} \text{ or } \mathbf{a} = \mathbf{C}^{-1}\mathbf{A}$$

where

$$\mathbf{C} = \begin{bmatrix} 1 & \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos\frac{M}{2}\omega_0 \\ 1 & \cos\omega_1 & \cdots & & \cos\frac{M}{2}\omega_1 \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & \cos\omega_{M/2} & \cdots & & \cos\frac{M}{2}\omega_{M/2} \end{bmatrix}$$

$$\mathbf{a} = [a[0]\ a[1]\ \cdots\ a[M/2]]^t$$

and

$$\mathbf{A} = \left[A(e^{j\omega_0})\ A(e^{j\omega_1})\ \cdots\ A(e^{j\omega_{M/2}})\right]^t$$

- In particular choose $M = 14$ ($M+1 = 15$) and let $A(e^{j\omega})$ be specified at equally spaced frequencies

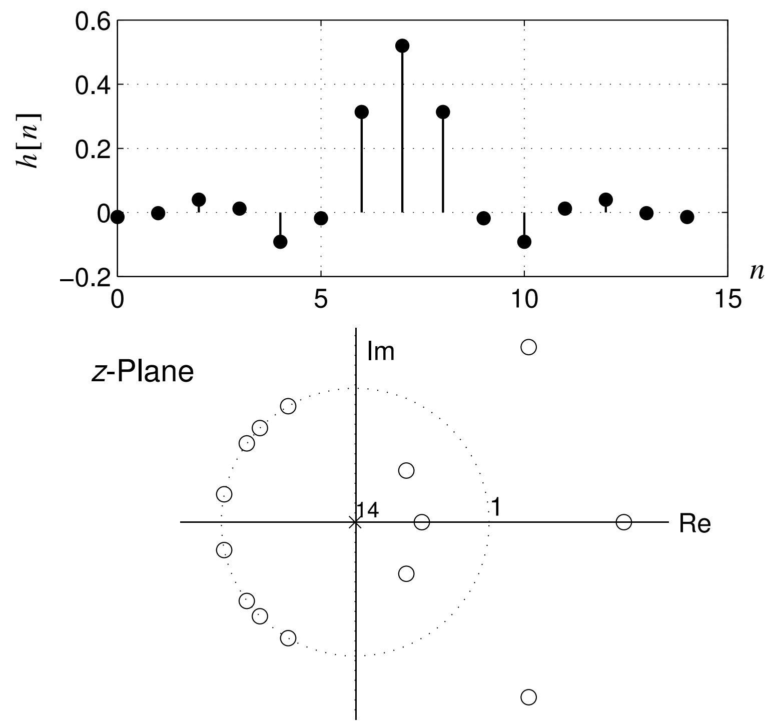
$$\omega_n = \frac{2\pi n}{M+1} = \frac{2\pi n}{15},\ n = 0,1,\ldots,M/2$$

- Choose the following frequency sample values:

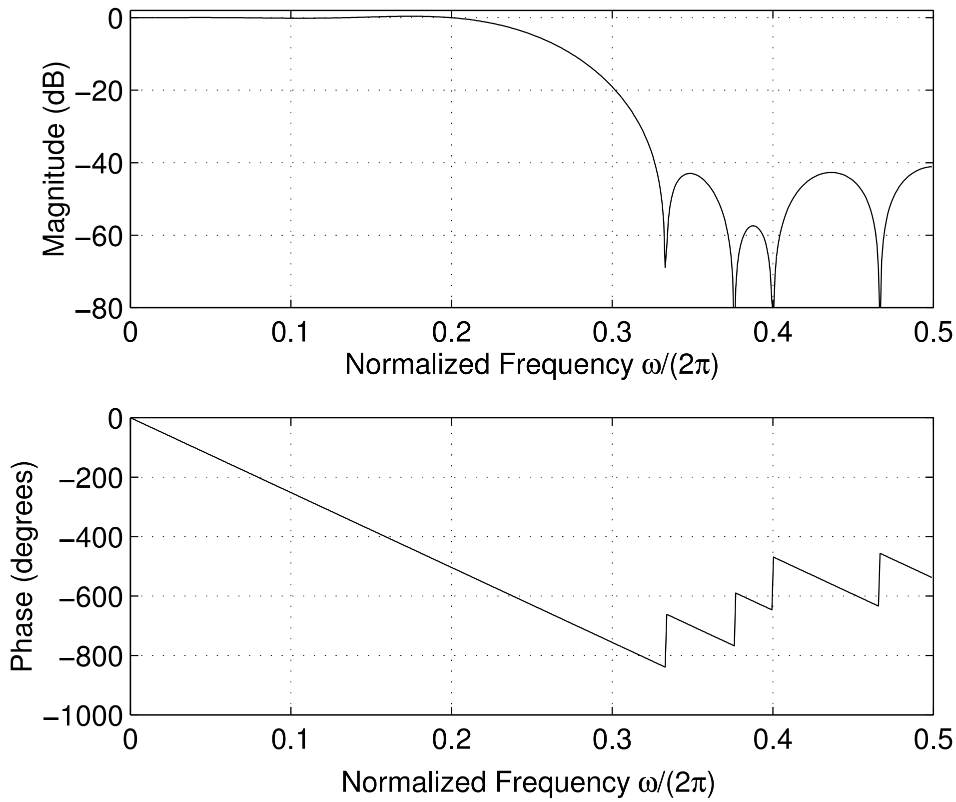
$$A(e^{j2\pi k/15}) = \begin{cases} 1, & k = 0,1,2,3 \text{ (passband)} \\ 0.4, & k = 4 \text{ (transition band)} \\ 0, & k = 5,6,7 \text{ (stopband)} \end{cases}$$

- Upon solving the system of equations we obtain

$$\begin{array}{ll} a[0] = 0.52 & a[1] = 0.6266352 \\ a[2] = -0.03617972 & a[3] = -0.18277604 \\ a[4] = 0.02446908 & a[5] = 0.08000008 \\ a[6] = -0.00389062 & a[7] = -0.02825786 \end{array}$$



Impulse response and Pole-zero plot



Plot of $|H(e^{j\omega})|_{db}$ for the $M = 14$ Type I FIR Filter

5.7.4 Relating Linear-Phase FIR Systems to Minimum-Phase Systems

- We know that linear-phase FIR systems have conjugate symmetry relationships between the zeros for those zeros not on the unit circle

- The zeros inside the unit circle correspond to a minimum-phase subsystems while those that lie outside the unit circle correspond to a maximum-phase subsystem

- An appropriate decomposition of a linear-phase FIR system is thus

$$H_{\text{lin}}(z) = H_{\text{min}}(z)H_{\text{uc}}(z)H_{\text{max}}(z)$$

  where due to the conjugate symmetry requirement for linear-phase

$$H_{\text{max}}(z) = H_{\text{min}}(z^{-1})z^{-M_i}$$

  and M_i is the number of minimum-phase zeros

- The system function $H_{\text{uc}}(z)$ represents the zeros that lie on the unit circle, of which we assume number M_0

- The system order is $M = 2M_i + M_0$

Example 5.21: A MATLAB Equal-ripple Design

- Consider a $M = 20$ design using the MATLAB function `remez()` which designs equal ripple FIR filters

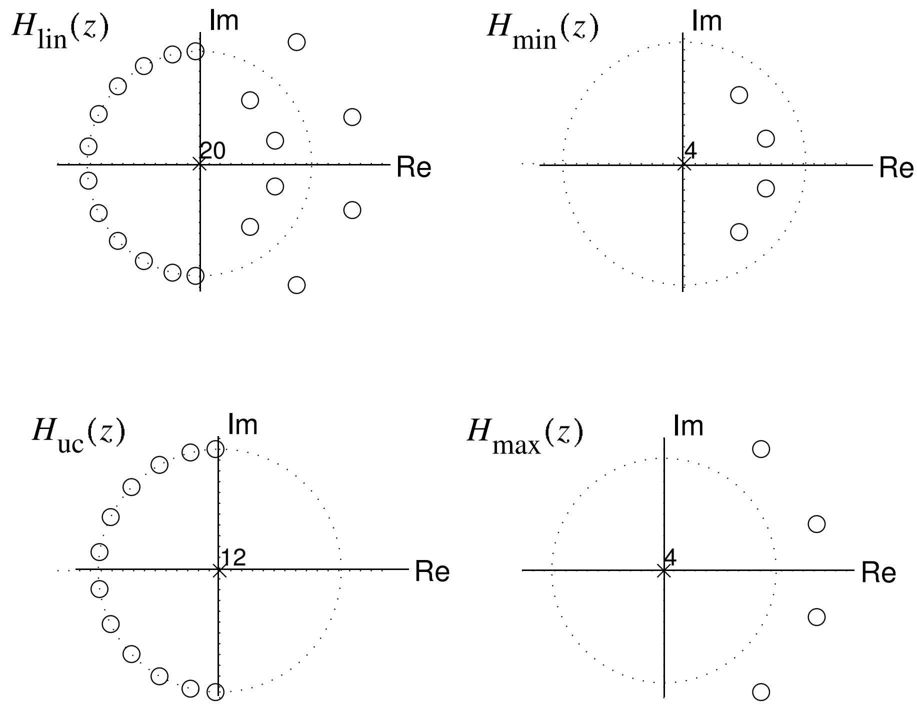
```
>> help remez % Truncated listing

 REMEZ Parks-McClellan optimal equiripple FIR filter design.
    B=REMEZ(N,F,A) returns a length N+1 linear phase (real, symmetric
    coefficients) FIR filter which has the best approximation to the
    desired frequency response described by F and A in the minimax
    sense. F is a vector of frequency band edges in pairs, in ascending
    order between 0 and 1. 1 corresponds to the Nyquist frequency or half
    the sampling frequency. A is a real vector the same size as F
    which specifies the desired amplitude of the frequency response of the
    resultant filter B. The desired response is the line connecting the
    points (F(k),A(k)) and (F(k+1),A(k+1)) for odd k; REMEZ treats the
    bands between F(k+1) and F(k+2) for odd k as "transition bands" or
    "don't care" regions. Thus the desired amplitude is piecewise linear
    with transition bands. The maximum error is minimized.

    B=REMEZ(N,F,A,W) uses the weights in W to weight the error. W has one
```

```
entry per band (so it is half the length of F and A) which tells
REMEZ how much emphasis to put on minimizing the error in each band
relative to the other bands.

B=REMEZ(N,F,A,'Hilbert') and B=REMEZ(N,F,A,W,'Hilbert') design filters
that have odd symmetry, that is, B(k) = -B(N+2-k) for k = 1, ..., N+1.
A special case is a Hilbert transformer which has an approx. amplitude
of 1 across the entire band, e.g. B=REMEZ(30,[.1 .9],[1 1],'Hilbert').

B=REMEZ(N,F,A,'differentiator') and B=REMEZ(N,F,A,W,'differentiator')
also design filters with odd symmetry, but with a special weighting
scheme for non-zero amplitude bands. The weight is assumed to be equal
to the inverse of frequency times the weight W. Thus the filter has a
much better fit at low frequency than at high frequency. This designs
FIR differentiators.

....more

See also CREMEZ, FIRLS, FIR1, FIR2, BUTTER, CHEBY1, CHEBY2, ELLIP,
FREQZ, FILTER.
```

- Here we choose a design with passband in the normalized frequency axis from 0 to 0.2 and stopband from 0.25 to 0.5
- The passband gain is set to unity and the stopband gain is set to zero

```
>> % Note remez assumes frequency normalization of 1 at pi
>> h = remez(20,2*[0 .2 .25 .5],[1 1 0 0]);
```

- The resulting filter is then decomposed in MATLAB into minimum-phase, unit circle, and maximum-phase zeros
- The pole-zero plots of $H_{\text{lin}}(z)$, $H_{\text{min}}(z)$, $H_{\text{uc}}(z)$, and $H_{\text{max}}(z)$ are shown below



Linear-phase pole-zero plots for $M = 20$ following decomposition

•