Chapter 4

Introduction to Random Processes (Z&T 7th ch 7)

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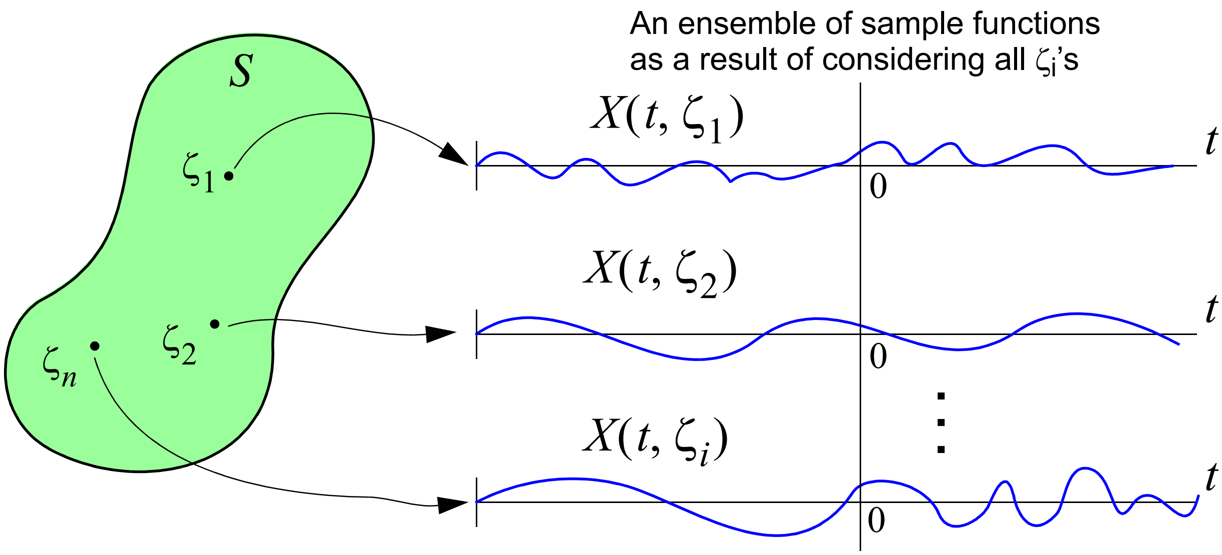
4.1 Introduction

The subject of random processes is a course within itself, thus this section will only introduce/review the basic concepts of random processes and in particular those areas of random process theory that are needed in digital communications.

4.2 Definition

A random process (RP) is most easily thought of as a family of functions $X(t, \zeta)$ such that each outcome ζ_i in a corresponding probability space, $\mathcal{S}$, is assigned a deterministic time function $X(t, \zeta_i)$.



From sample space to time functions

4.2.1 Sample Functions and Ensembles

- The notation $X(t)$ is used to represent an RP, but actually carries one of four interpretations:

  1. A family of time functions; both t and ζ variables
  2. A single time function; t a variable and ζ fixed
  3. A random variable; t fixed and ζ a variable
  4. A single number; t fixed and ζ fixed

- An alternative view of a random process is to view it as a collection of random variables (rvs) by writing

$$\{X(t_1), X(t_2), X(t_3)\ldots\}$$

  where the index set is the real numbers if we have a continuous-time RP, and the integers if we have a discrete-time RP.

4.2.2 RP Descriptions

- A *complete statistical description* requires the joint probability density function (pdf) of $X(t)$ be known for any integer n and any choice of $(t_1, t_2, t_3, \ldots, t_n) \in P^n$, i.e.,

$$f_{X_1,X_2,\ldots,X_n}(x_1, t_1; x_2; t_2; \ldots; x_n, t_n)$$

- A complete description is typically difficult to obtain and not required

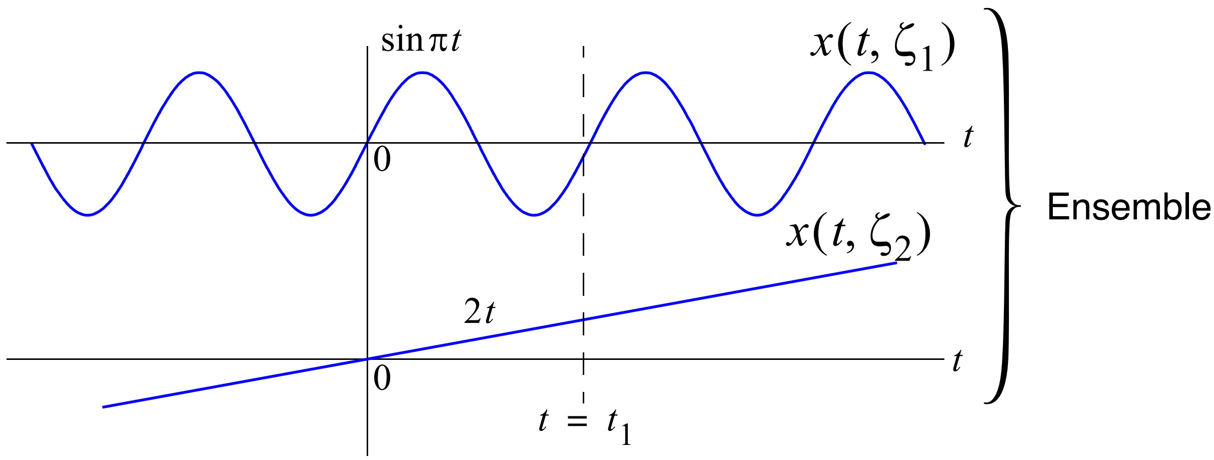
- First ($n = 1$) and second ($n = 2$) order statistics are often all that is required

Example 4.1: Two Sample Function Ensemble

- Consider the process

$$X(t) = \begin{cases} \sin \pi t, & \text{with probability } 1/2 \\ 2t, & \text{with probability } 1/2 \end{cases}$$

- Find the first-order density $f_X(x, t)$



The ensemble

- By inspection we can write that for any sampling time

$$f_X(x, t) = \frac{1}{2}\delta(x - \sin \pi t) + \frac{1}{2}\delta(x - 2t)$$

Example 4.2: Random Exponential Signal

- An ensemble of exponential pulses with random time constants

$$X(t) = e^{-\mu t}\, u(t)$$

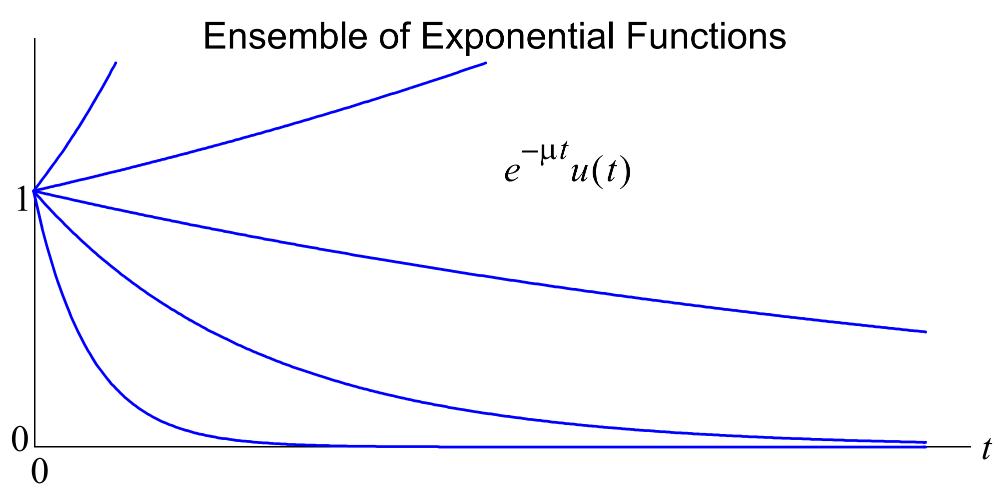
where μ is an rv with pdf $f_\mu(\mu)$

- Find the first-order density $f_X(x,t)$
- Using a simple function of a single rv transformation we can write

$$\text{transform equation}\quad \begin{cases} x = e^{-\mu t} \Rightarrow \mu_1 = -\frac{1}{t}\ln x, x \geq 0 \\ \frac{dx}{d\mu} = -te^{-\mu t} = -te^{\ln x} = -tx \end{cases}$$

- So

$$f_X(x,t) = \frac{1}{|tx|} f_\mu\left(-\frac{\ln x}{t}\right) u(t)u(x)$$



The ensemble

Example 4.3: **Randomly Phased Sinusoid**

- A randomly phased sinusoid

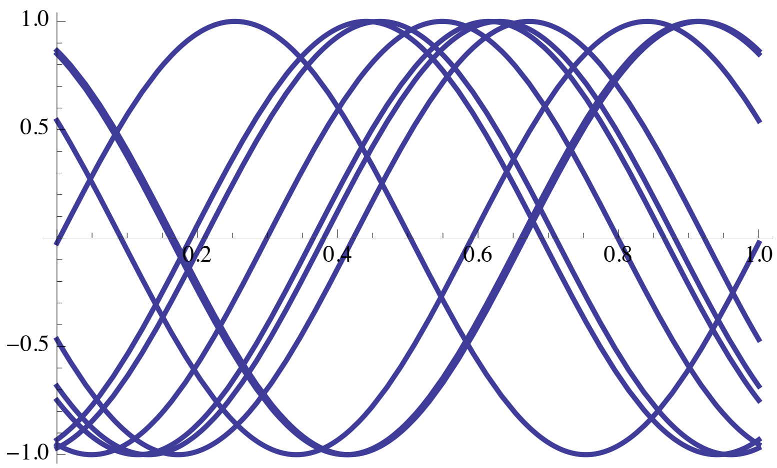
$$X(t) = A\cos(2\pi f_0 t + \Phi)$$

where Φ is a random variable uniformly distributed on $[0, 2\pi)$.

```
X3 = A Cos[2 π f0 t + ϕ];
Phase = Table[2 π Random[], {k, 1, 10}]
```

```
{3.51542, 2.32311, 3.39102, 4.68982, 2.83274,
 2.41771, 0.555394, 0.525564, 2.05838, 0.998119}
```

```
Plot[Table[X3 /. {A → 1, f0 → 1, ϕ → Phase[[k]]}, {k, 1, 10}],
 {t, 0, 1}, PlotStyle → Thick]
```



Ten sample functions for $A = 1$ and $f_0 = 1$

4.2.3 Statistical Averages

Mean

- The mean is the deterministic function defined by

$$m_X(t) = E[X(t)] = \overline{X(t)} = \int_{-\infty}^{\infty} x f_X(x,t)\, dx$$

Note: Often times we can more simply use the fundamental theorem of expectation to find $m_X(t)$

Variance

- The variance is the deterministic function defined by

$$
\begin{aligned}
\sigma_X^2(t) &= E\left\{[X(t) - m_X(t)]^2\right\} = \overline{X^2(t)} - \overline{X(t)}^2 \\
&= \int_{-\infty}^{\infty} [x(t) - m_x(t)]^2 f_X(x,t)\, dx
\end{aligned}
$$

Covariance

- The covariance between two rv is extended to a random process as

$$
\begin{aligned}
\mu_X(t, t+\tau) &= E\left[[X(t) - m_X(t)][X(t+\tau) - m_X(t+\tau)]\right] \\
&= E[X(t)X(t+\tau)] - m_X(t)m_X(t+\tau)
\end{aligned}
$$

Autocorrelation Function

- The autocorrelation function is the deterministic function defined by

$$
\begin{aligned}
R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, t_1; x_2, t_2)\, dx_1 dx_2
\end{aligned}
$$

Example 4.4: **Randomly Phased Sinusoid part – 2**

- Consider the randomly phased sinusoid and find its mean, variance, and autocorrelation

- Mean: using the fundamental theorem of expectation we can write

$$
\begin{aligned}
m_X(t) &= E[g(\Phi,t)] = E[A\cos(2\pi f_0 t + \Phi)] \\
&= \int_{-\infty}^{\infty} A\cos(2\pi f_0 t + \phi)\, f_\Phi(\phi)\, d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} A\cos(2\pi f_0 t + \phi)\, d\phi \\
&= 0
\end{aligned}
$$

- Variance:

$$
\begin{aligned}
\sigma_X^2(t) &= E\big[A^2\cos^2(2\pi f_0 t + \Phi)\big] \\
&= \frac{A^2}{2}\int_0^{2\pi}\big[1 + \cos(4\pi f_0 t + 2\phi)\big]\frac{d\phi}{2\pi} \\
&= \frac{A^2}{2} \quad \text{familiar?}
\end{aligned}
$$

- Autocorrelation

$$
\begin{aligned}
R_{XX}(t+\tau,\tau) &= E\big[A\cos(2\pi f_0(t+\tau) + \Phi) \\
&\qquad \cdot A\cos(2\pi f_0 t + \Phi)\big] \\
&= \frac{A^2}{2}\int_0^{2\pi}\big[\cos(2\pi f_0\tau) \\
&\qquad + \cos(2\pi f_0(2t+\tau) + 2\phi)\big]\frac{d\phi}{2\pi} \\
&= \frac{A^2}{2}\cos(2\pi f_0\tau) \overset{\text{also}}{=} R_{XX}(\tau)
\end{aligned}
$$

Note: $\sigma_X^2 = R_{XX}(t,t) - m_X^2(t)$

4.2.4 Stationarity

Strict Sense

A process is stationary in the strict sense if its statistics are not changed when the time origin changes, in particular

$$
\begin{aligned}
f_X(x,t) &= f_X(x) \Rightarrow m_X(t) = m_x,\ \ \sigma_X^2(t) = \sigma_X^2 \\
f_{X_1X_2}(x_1,t_1;x_2,t_2) &= f_{X_1X_2}(x_1,x_2;t_1-t_2) \\
&\Rightarrow R_{XX}(t+\tau,t) = R_{XX}(\tau)
\end{aligned}
$$

Wide Sense

A process is wide-sense stationary if the following is satisfied:

1. $m_X(t) = E[X(t)] = m_x$ is independent of t
2. $R_{XX}(t_1,t_2) = R_{XX}(t_1-t_2) = R_{XX}(\tau)$ depends only on the time difference, and not on the time origin

- The randomly phased sinusoid is an example of a wide sense stationary (WSS) RP

Cyclostationary

A process is wide sense cyclostationary if both the mean and autocorrelation function are periodic in t with some period T_0, i.e.,

1. $m_X(t+kT_0) = m_X(t)$
2. $R_{XX}(t+\tau+kT_0, t+kT_0) = R_{XX}(t+\tau,t)$

Example 4.5: A Product Process

- Suppose a process $Y(t)$ is formed from the product of a WSS RP $X(t)$ and a sinusoid

$$Y(t) = X(t)\cos(2\pi f_0 t)$$

  Note: No random phase this time

- Mean

$$m_Y(t) = E\big[X(t)\cos(2\pi f_0 t)\big] = m_X \cos(2\pi f_0 t)$$

- Autocorrelation

$$\begin{aligned} R_{YY}(t+\tau, t) &= E\big[X(t+\tau)\cos(2\pi f_0 (t+\tau)) \\ &\qquad \cdot X(t)\cos(2\pi f_0 t)\big] \\ &= R_{XX}(\tau) \\ &\qquad \cdot \frac{1}{2}\big[\cos(2\pi f_0 \tau) + \cos(2\pi f_0 (2t+\tau))\big] \end{aligned}$$

- From both the mean and autocorrelation being periodic, we see that the process is widesense cyclostationary

4.2.5 Ergodicity

- A random process is *ergodic* if time and ensemble averages are interchangeable
- A detailed mathematical discussion of ergodic theory, yet still within an electrical engineering context can be found in the text by Gray and Davisson[1]
- **Define**: Ensemble Average

$$E[g(X(t_0))] = \int_{-\infty}^{\infty} g(x) f_X(x, t_0)\, dx$$

- **Define**: Time Average (for a single sample function)

$$\langle g(x) \rangle_i = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} g(x(t, \zeta_i))\, dt$$

- Note: The time average is in general a random variable, but tends to a constant with respect to the particular RP realization chosen for the case of an ergodic process

Mean Ergodic

An RP is *mean ergodic* if

$$m_X(t) = E[X(t)] = \langle x(t) \rangle = \text{constant} = m_x$$

[1] R. M. Gray and L. D. Davisson, *Random Processes: A Mathematical Approach for Engineers*, Prentice Hall, New Jersey, 1986.

Autocorrelation Ergodic

An RP is *autocorrelation ergodic* if

$$\begin{aligned} R_{XX}(t+\tau, t) &= E[X(t+\tau)X(t)] \\ &= \langle x(t+\tau)x(t)\rangle \\ &= R_{XX}(\tau) \end{aligned}$$

Example 4.6: **Randomly Phased Sinusoid part – 3**

- Again we consider

$$X(t) = A\cos(2\pi f_0 t + \Phi)$$

- Time average mean:

$$\langle A\cos(2\pi f_0 t + \phi)\rangle = 0$$

- Time average autocorrelation function:

$$\begin{aligned} &\langle A\cos(2\pi f_0(t+\tau) + \phi)A\cos(2\pi f_0 t + \phi)\rangle \\ &\quad = \frac{A^2}{2}\big[\langle \cos(2\pi f_0 \tau\rangle + \langle \cos(2\pi f_0(2t+\tau) + 2\phi)\rangle\big] \\ &\quad = \frac{A^2}{2}\cos(2\pi f_0 \tau) \end{aligned}$$

Time Average Comparisons

Interpretation of statistical averages for ergodic processes as measurable quantities:

1. $E[X(t)] = \langle x(t) \rangle =$ dc component
2. $E[X(t)]^2 = \langle x(t) \rangle^2 =$ dc power
3. $E[X^2(t)] = \langle x^2(t) \rangle = P_X =$ total power
4. $\sigma_X^2 = E[X^2(t)] - E^2[X(t)] = \langle x^2(t) \rangle - \langle x(t) \rangle^2 =$ ac power

4.3 Multiple Random Processes

When a random process is passed through a system a new or output RP is generated. It reasonable then that one would consider the correlation between the input and output RP's, or for that matter the statistical dependence between any pair of random signals, say signal and noise.

4.3.1 Crosscorrelation

For two RP's $X(t)$ and $Y(t)$ we define the crosscorrelation as

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = R_{YX}(t_2, t_1)$$

- If X and Y are uncorrelated processes, then

  $$R_{XY}(t_1, t_2) = m_x(t_1)m_Y(t_2)$$

  which follows from the definition of uncorrelated random variables

- X and Y are jointly WSS implies that

$$R_{XY}(t+\tau,t) = R_{XY}(\tau)$$

Example 4.7: **Sum RP**

- Find the autocorrelation function of the process

$$Z(t) = X(t) + Y(t)$$

assuming that X and Y are jointly WSS

- By definition

$$\begin{aligned} R_{ZZ}(\tau) &= E[Z(t+\tau)Z(t)] \\ &= E\big[[X(t+\tau)+Y(t+\tau)][X(t)+Y(t)]\big] \\ &= E\big[X(t+\tau)X(t)\big] + E\big[X(t+\tau)Y(t)\big] \\ &\quad + E\big[Y(t+\tau)X(t)\big] + E\big[Y(t+\tau)Y(t)\big] \\ &= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau) \end{aligned}$$

- If the processes are also uncorrelated and one of them has zero mean, then

$$R_{XY}(\tau) = m_x m_y = 0 = R_{YX}(\tau)$$

Note: X and Y are now said to be orthogonal processes

- Finally, under the uncorrelated assumption

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)$$

4.4 Power Spectra

The frequency domain representation of a random process requires the use for the Fourier transform, but we must be careful to insure that the Fourier transform exists in a meaningful way.

- We start by defining a truncated sample function
$$X_T(t,\zeta_i) = \begin{cases} x(t,\zeta_i), & |t| < T/2) \\ 0, & \text{otherwise} \end{cases}$$
- The energy spectral density of a particular truncated sample function is
$$\left|X_{T_i}(f)\right|^2 = \left|\mathcal{F}\{x_T(t,\zeta_i)\}\right|^2$$
- The power spectral density of a particular sample function is $|X_{T_i}(f)|^2/T$ as T becomes large, with units of watts/Hz
- The ensemble average of the above results in the power spectrum being defined as
$$S_x(f) = \lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T}$$
- The above definition does not always provide an easy means of obtaining the power spectrum of a given RP

4.4.1 Wiener-Kinchine Theorem

- The Wiener-Kinchine theorem provides a linkage between the power spectrum and autocorrelation function via the Fourier transform
- The formal theorem statement is for any finite τ and interval U of length $|\tau|$, that gives

$$\left|\int_U R_{XX}(t+\tau,t)\,dt\right| < \infty,$$

then the power spectrum of $X(t)$ is

$$S_x(f) = \mathcal{F}\big\{\langle R_{XX}(t+\tau,t)\rangle\big\}$$

- In practical applications it is the corollaries to this theorem that are of the most significance
- For details on the proofs see Proakis and Salehi[2].
- **Corollary**: If $X(t)$ is WSS and $|\tau R_{XX}(\tau)| < \infty$ for all τ, then

$$\boxed{S_x(f) = \mathcal{F}\big\{R_{XX}(\tau)\big\} \stackrel{\text{also}}{=} \mathcal{F}\{R_X(\tau)\}}$$

  - This result is perhaps the most important result for this course, i.e., the autocorrelation and power spectrum are a Fourier transform pair

[2] J. G. Proakis and M. Salehi, *Commnications Systems Engineering*, Prentice Hall, 1994

Example 4.8: Randomly Phased Sinusoid – part 4

- For $x(t) = A\cos(2\pi f_0 t + \phi)$ we know that

$$R_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$

- The power spectral density (PSD) is now

$$S_x(f) = \mathcal{F}\{R_{XX}(\tau)\} = \frac{A^2}{4}\big[\delta(f + f_0) + \delta(f - f_0)\big]$$

- **Corollary**: If $X(t)$ is cyclostationary (in the wide sense) and

$$\left|\int_0^{T_0} R_X(t+\tau, t)\,dt\right| < \infty$$

then

$$S_x(f) = \mathcal{F}\underbrace{\left\{\frac{1}{T_0}\int_0^{T_0} R_X(t+\tau, t)\,dt\right\}}_{\overline{R_X(\tau)}}$$

Example 4.9: Modulated Carrier Process – part 2

- Revisit the cyclostationary process

$$Y(t) = X(t)\cos(2\pi f_0 t)$$

where $X(t)$ is WSS

- The power spectrum is

$$
\begin{aligned}
S_Y(f) &= \mathcal{F}\{\overline{R_Y(\tau)}\} \\
&= \mathcal{F}\left\{\frac{1}{T_0}\int_0^{T_0} R_X(\tau)\frac{1}{2}[\cos(2\pi f_0\tau) \right. \\
&\qquad \left. + \cos(2\pi f_0(2t+\tau))]\,dt\right\} \\
&= \mathcal{F}\left\{\frac{R_X(\tau)}{2}\cos(2\pi f_0\tau)\right\} \\
&= \frac{1}{4}\big[S_x(f+f_0) + S_x(f-f_0)\big]
\end{aligned}
$$

WSS RP Autocorrelation Function Properties

If RP $X(t)$ is WSS then its autocorrelation function has the following properties:

1. $R_X(0) = E[X^2(t)] \geq |R_X(\tau)|$ for all τ; this implies that a relative maximum of $|R_X(\tau)|$ exists at $\tau = 0$

   proof – Consider

   $$|X(t) \pm X(t+\tau)|^2 \geq 0$$

   Taking the expectation of each term we have

   $$E[X^2(t)] \pm 2E[X(t+\tau)X(t)] + E[X^2(t+\tau)] \geq 0$$

   from which we establish that

   $$2R_X(0) \pm 2R_X(\tau) \geq 0 \quad \text{or} \quad -R_X(0) \leq R_X(\tau) \leq R_X(0)$$

2. $R_X(-\tau) = E[X(t-\tau)X(t)] = R_X(\tau)$; an even function of τ.

   proof – From the definition we have that

$$\begin{aligned} R_X(\tau) &= E[X(t+\tau)X(t)] = E[X(t)X(t+\tau)] \\ &= R_X(t-(t+\tau)) = R_X(-\tau) \end{aligned}$$

3. $\lim_{|\tau|\to\infty} R_X(\tau) = E[X(t)]^2$ if $X(t)$ does not contain periodic components.

   proof: For large τ $X(t+\tau)$ and $X(t)$ decorrelate making

$$\lim_{|\tau|\to\infty} R_X(\tau) \simeq E[X(t+\tau)]E[X(t)] \overset{\text{WSS}}{=} E[X(t)]^2$$

4. If for some T_0, $R_X(T_0) = R_X(0)$, then it can be shown that for any integer k, $R_X(kT_0) = R_X(0)$.

   proof: Given that $X(t)$ is periodic, as in the randomly phase sinusoid, the integrand in computing $R_X(\tau)$ is periodic so $R_X(\tau)$ is periodic; also similar to cyclostationary process properties

5. The PSD is nonnegative, i.e., $S_x(f) \geq 0$ for all values of f.

   proof: The non-negativity of $S_x(f)$ follows from the PSD definition

$$S_x(f) = \lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T}$$

Note: If $X(t)$ is ergodic, then above properties also hold in the time average sense.

4.4.2 Random Pulse Train Autocorrelation Function

- In the study of digital communications we are very interested in the power spectra of data waveforms
- One such classical example is the random pulse train

$$X(t) = \sum_{k=-\infty}^{\infty} a_k p(t - kT - \Delta)$$

where T is the bit period, a_k is a sequence of random variables such that

$$E[a_{k+m}a_k] = R_m$$

$p(t)$ is a deterministic pulse shape function, and Δ uniform on $(-T/2, T/2)$, establishes a random epoch for the arrival of the pulse train edges

  – If we omit Δ we have a cyclostationary process, whereas including Δ insures that $X(t)$ is WSS

- To obtain the PSD of $X(t)$ we begin by finding $R_X(\tau)$
  – Initially write $X(t+\tau)X(t) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_n a_k \cdots$, but equivalently let $n = k + m$ works and $-\infty < m < \infty$

$$R_X(\tau) = E[X(t+\tau)X(t)] = E\Bigg[\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_{k+m}a_k \\ p\big(t + \tau - (k+m)T - \Delta\big)p\big(t - kT - \Delta\big)\Bigg]$$

- By assumption the a_k are independent of Δ so we can make some simplifications

$$
\begin{aligned}
R_X(\tau) &= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} E[a_{k+m}a_k] \\
&\qquad E\big[p(t+\tau-(k+m)T-\Delta)p(t-kT-\Delta)\big] \\
&= \sum_{m=-\infty}^{\infty} R_m \sum_{k=-\infty}^{\infty} \int_{-T/2}^{T/2} p\big(t+\tau-(k+m)T-\Delta\big) \\
&\qquad p\big(t-kT-\Delta\big)\frac{d\Delta}{T}
\end{aligned}
$$

- To reduce the double integral we change variables by letting $u = t - kT - \Delta$, $dU = -d\Delta$, etc.,

$$
\begin{aligned}
R_X(\tau) &= \sum_{m=-\infty}^{\infty} R_m \\
&\qquad \sum_{k=-\infty}^{\infty} \int_{t-(k+1/2)T}^{t-(k-1/2)T} p(u+\tau-mT)p(u)\frac{du}{T} \\
&= \sum_{m=-\infty}^{\infty} R_m \left[\frac{1}{T}\int_{-\infty}^{\infty} p(u+\tau-mT)p(u)\,du\right]
\end{aligned}
$$

- The term inside the square brackets can be defined as the deterministic pulse autocorrelation function

$$
r_p(\tau) \triangleq \frac{1}{T}\int_{-\infty}^{\infty} p(t+\tau)p(t)\,dt
$$

- Finally,

$$
\boxed{R_X(\tau) = \sum_{m=-\infty}^{\infty} R_m r_p(\tau - mT)}
$$

and

$$S_x(f) = \sum_{m=-\infty}^{\infty} R_m R_p(f) e^{-j2\pi m f T}$$

where $R_p(f) = \mathcal{F}\{r_p(\tau)\}$

- For the special case where the data bits a_k are uncorrelated, R_m is zero everywhere except when $m = 0$, then we have

$$R_X(\tau) = R_0 r_p(\tau) \quad \text{and} \quad S_x(f) = R_0 R_p(f)$$

Example 4.10: **Random Pulse Train Simulation**

- In this example a random pulse train waveform simulation is created in MATLAB
- We then estimate the autocorrelation function and power spectrum
- The function `PT_mod()` generates a discrete-time pulse train waveform

```
function [x,data] = PT_mod(Nbit,Ns,shape)
% x = PT_mod(Nbit,Ns,shape)
% Random binary pulse train waveform simulation
%///////////////////////////////////////////////////////////////////////
%  Nbit = number of bits to simulate
%    Ns = number of samples per bit
% shape = full response shape type, rectangular or 'REC',
%         raised cosne or 'RC', and half-sine or 'HS'
%     x = the output waveform
%  data = the data bits consisting of +1/-1 values at one sample/bit
% //////////////////////////////////////////////////////////////////////
%
% Mark Wickert September 2010
```

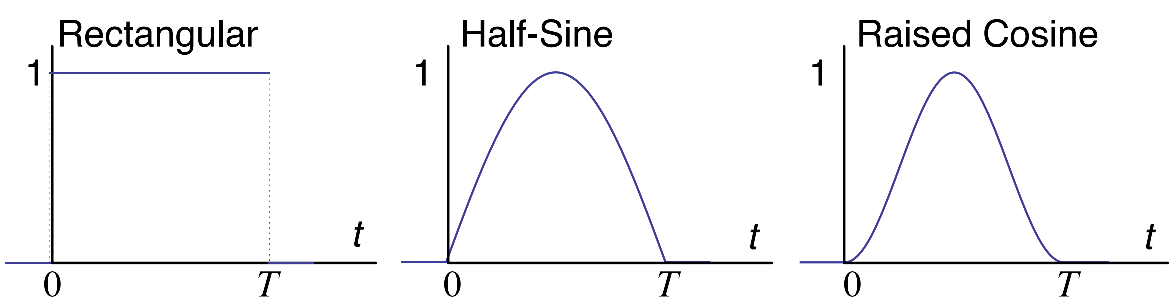
```
data = 2*randi(2,1,Nbit)-3; % random integers on [1,2] scaled to +1/-1
imp_mod = [data; zeros(Ns-1,Nsbit)];
imp_mod = reshape(imp_mod,1,Ns*Nbit);

n = 0:Ns-1;

switch upper(shape)
    case 'REC'
        P = ones(1,Ns);
    case 'RC'
        P = (cos(2*pi*(n-fix(Ns/2))/Ns)+1)/2;
    case 'HS'
        P = sin(2*pi*n/(2*Ns));
    otherwise
        error('Unknown shape')
end

x = filter(P,1,imp_mod);
```

- The data bits take on values of ± 1, but could be scaled to any value, e.g., $\pm A$
- Three pulse shapes are implemented:



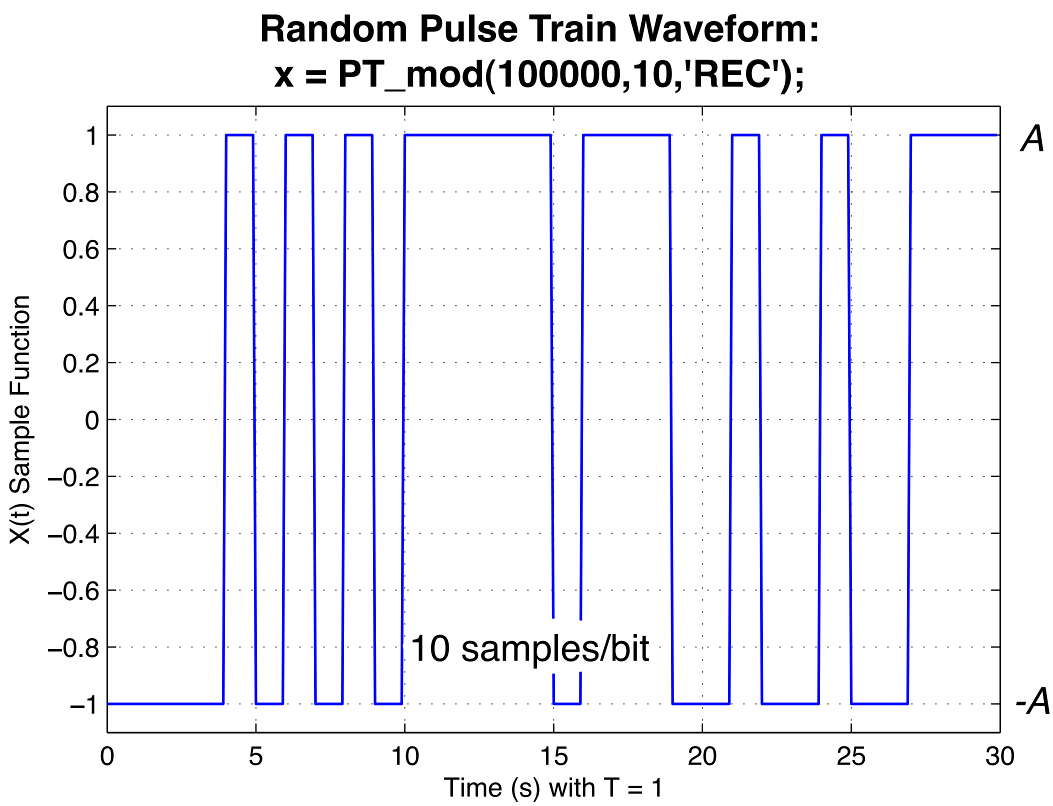
$$p_{\text{rec}}(t) = \Pi\left(\frac{t - \frac{T}{2}}{T}\right) \qquad \sin\left(\frac{2\pi t}{2T}\right) p_{\text{rec}}(t) \qquad \frac{1 + \cos\left(\frac{2\pi(t-T/2)}{T}\right)}{2} p_{\text{rec}}(t)$$

Pulse shapes $p(t)$ used in m-code

- To generate 100,000 bits using the REC pulse shape we enter:

```
>> x = PT_mod(100000,10,'REC');
```

```
>> t = [0:length(x)-1]/10;
>> plot(t(1:300),x(1:300))
>> axis([0 30 -1.1 1.1])
>> grid
>> xlabel('Time (s) with T = 1')
>> ylabel('X(t) Sample function')
>> ylabel('X(t) Sample Function')
>> print -tiff -depsc time_plot.eps
```



One sample function of $X(t)$ with REC pulse

- Under the REC pulse shape, and assuming $\pm A$ amplitude pulses, we have

$$E[a_{k+m}a_k] = A^2\delta[m] = A^2\delta_m$$

- With $p(t) = \Pi\big((t - T/2)/T\big)$ we have

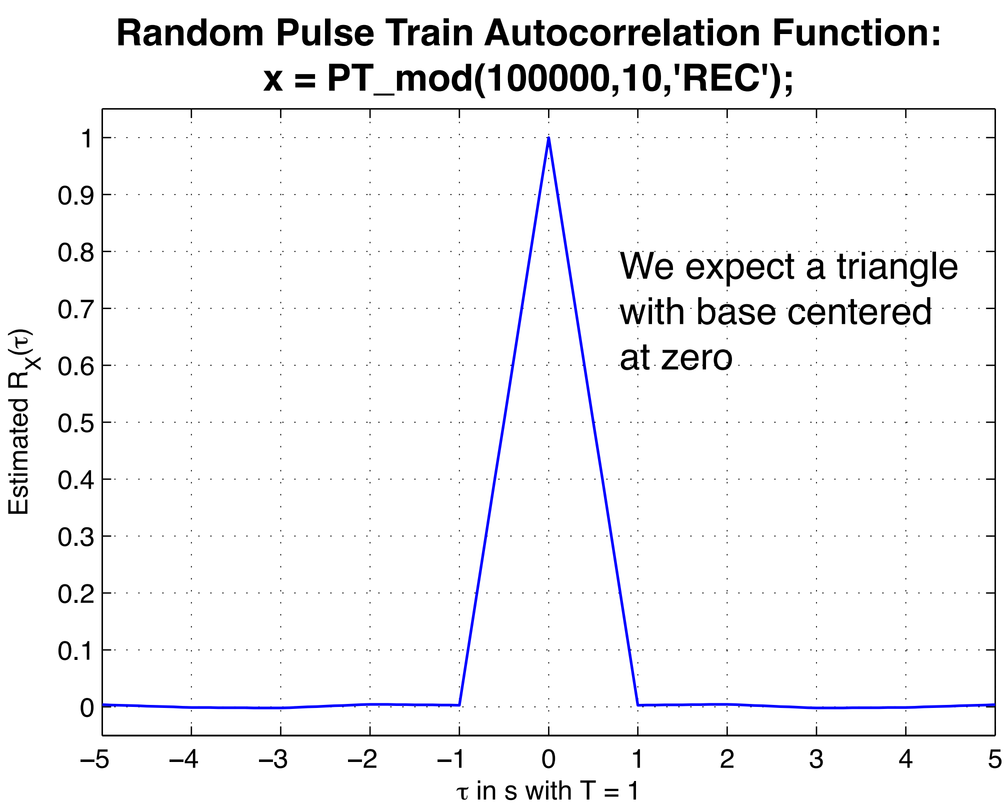
$$r_p(\tau) = \Lambda\left(\frac{\tau}{T}\right)$$

which is a triangle shape of height one and base width of $2T$, thus

$$R_X(\tau) = A^2\Lambda\left(\frac{\tau}{T}\right)$$

- We can estimate $R_X(\tau)$ using the MATLAB function `xcorr`

```
>> [Rx,lags] = xcorr(x,50,'unbiased');
>> plot(lags/10,Rx)
>> grid
>> xlabel('tau in s with T = 1')
>> ylabel('Estimated R_X(tau)')
>> print -tiff -depsc Rx_plot.eps
```



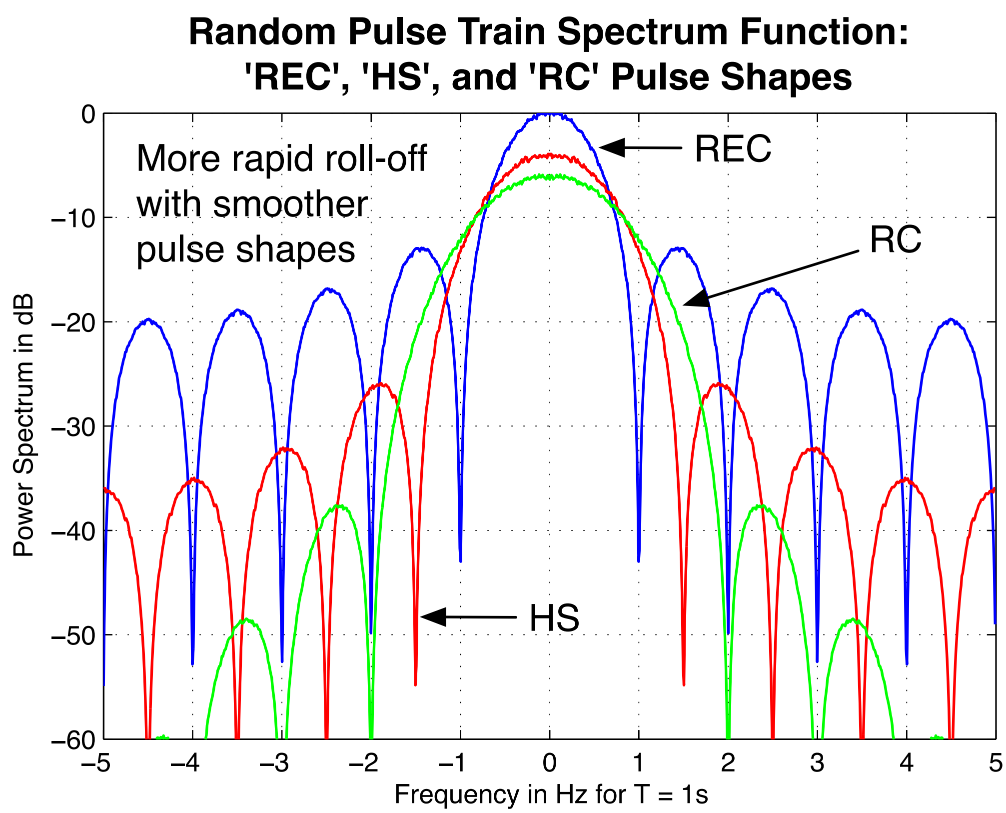
Estimated $R_X(\tau)$ with REC pulse

- The theoretical PSD for the REC pulse shape is

$$S_x(f) = \mathcal{F}\left\{A^2\Lambda\left(\frac{\tau}{T}\right)\right\} = A^2T\,\text{sinc}^2(fT)$$

- We can estimate $S_x(f)$ using the MATLAB function `pwelch()`
- Here we have provided a *wrapper* function that also obtains a two-sided PSD estimate and sets some values in `pwelch`

```
>> x_rec = PT_mod(100000,10,'REC');
>> x_hs = PT_mod(100000,10,'HS');
>> x_rc = PT_mod(100000,10,'RC');
>> spec_plot(x_rec,2^10,10);
>> hold
Current plot held
>> spec_plot(x_hs,2^10,10,'r');
>> spec_plot(x_rc,2^10,10,'g');
>> axis([-5 5 -60 0])
>> xlabel('Frequency in Hz for T = 1s')
>> print -tiff -depsc Sx_plot.eps
```



Estimated $S_x(f)$ comparing REC, HS, and RC pulse shapes

- The function `spec_plot` is documented below:

```
function [Px,F] = spec_plot(x,N,fs,color)
% spec_plot(x,N,fs,color)
% Plot the estimated power spectrum of real and complex baseband signals
% using the toolbox function pwelch(). This replaces the use of psd().
%
%     x = complex baseband data record, at least N samples in length
%     N = FFT length
%    fs = Sampling frequency in Hz
% color = line color and type as a string, e.g., 'r' or 'r--'
%
%////////// Optional output variables ///////////////////////////////////
% Px = power spectrum estimate values on two-sided frequency axis
%  F = the corresponding frequency axis values
%
% Mark Wickert, November 2007
% Modified September 2010

% The default window is Hamming with 50% overlap. Here we use the
% hanning window as the old psd.m function used this by default. The old
% psd function also used 0% overlap by default.
if isreal(x)
    [Px,F] = pwelch(x,hanning(N),50,N,fs,'two-sided');
else
    [Px,F] = pwelch(x,hanning(N),50,N,fs);
end

N = fix(length(F)/2);
F = [F(end-N+1:end)-fs; F(1:N)];
Px = [Px(end-N+1:end); Px(1:N)];

if nargout == 2
    return
end

if nargin == 3,
    plot(F,10*log10(Px))
else
    plot(F,10*log10(Px),color)
end

xlabel('Frequency (Hz)')
ylabel('Power Spectrum in dB')
grid
```

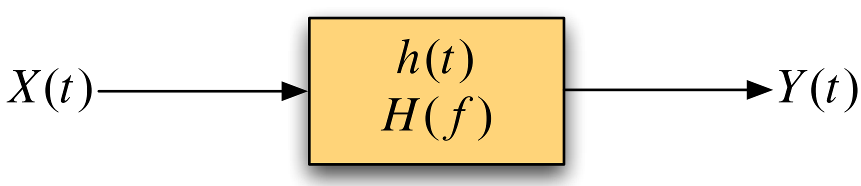
4.4.3 Cross-Power Spectral Density

- In an earlier example involving the sum of two random processes we encountered the cross-correlation function $R_{XY}(\tau)$
- The cross-power spectral density is the Fourier transform of $R_{XY}(\tau)$

$$S_{xy}(f) = \mathcal{F}\{R_{XY}(\tau)\}$$

4.5 Random Processes in LTI Systems

Here we will summarize the results of a WSS process passing through an LTI system.



LTI system with RP input

Mean

$$\begin{aligned} m_y &= E[X(t) * h(t)] = E[X(t)] * h(t) \\ &= m_x \int_{-\infty}^{\infty} h(t)\, dt = m_x H(0) \end{aligned}$$

Autocorrelation

$$
\begin{aligned}
R_Y(\tau) &= E[Y(t+\tau)Y(t)] \\
&= E\Big[\big(X(t+\tau) * h(t+\tau)\big)\big(X(t) * h(t)\big)\Big] \\
&= R_X(\tau) * h(\tau) * h(-\tau) = R_X(\tau) * \underbrace{\big[h(\tau) * h(-\tau)\big]}_{r_h(\tau)}
\end{aligned}
$$

- The proof involves writing out the convolution integrals in two places and then working through a change of variables, assuming of course that $X(t)$ is WSS

Cross Correlation

$$
\begin{aligned}
R_{XY}(\tau) &= E\big[X(t+\tau)Y(t)\big] = E\big[X(t+\tau)\big(X(t) * h(t)\big)\big] \\
&= R_X(\tau) * h(-\tau)
\end{aligned}
$$

Power Spectrum

$$
\begin{aligned}
S_y(f) &= \mathcal{F}\big\{R_X(\tau) * h(\tau) * h(-\tau)\big\} \\
&= S_x(f)H(f) \cdot H^*(f) = S_x(f)|H(f)|^2
\end{aligned}
$$

Cross Spectrum

$$
\begin{aligned}
S_{xy}(f) &= S_x(f)H^*(f) \\
S_{yx}(f) &= S_x(f)H(f)
\end{aligned}
$$

- **Note**: Both $S_x(f)$ and $S_y(f)$ are nonnegative real functions, but the cross spectrum is in general complex

Example 4.11: Signal plus Uncorrelated Noise

- Let

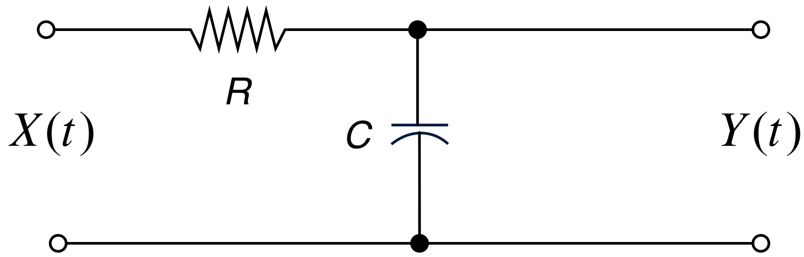
$$X(t) = S(t) + W(t)$$

where by assumption $R_{SW}(\tau) = 0$ and $m_w = 0$

- The power spectrum is simply

$$S_x(f) = S_s(f) + S_w(f)$$

- Suppose $X(t)$ is passed through an RC lowpass filter



RC lowpass circuit

- The output power spectrum is

$$S_y(f) = S_x(f)|H(f)|^2 = \left[S_s(f)+S_w(f)\right]\cdot\frac{1}{1+(2\pi fRC)^2}$$

4.6 Gaussian (Normal) Process

A process is Gaussian (Normal) if the PDF for all orders of the random process statistics are jointly Gaussian. For the case of $n = 2$

we have

$$f_{X_1X_2}(x_1, t_1; x_2, t_2) = \frac{1}{2\pi\sigma_X^2\sqrt{1-\rho^2}} \cdot \exp\left[-\frac{1}{2\sigma_X^2(1-\rho^2)}\left\{(x_1 - m_x)^2 - 2\rho(x_1 - m_x)(x_2 - m_x) + (x_2 - m_x)^2\right\}\right]$$

where $\rho = \left[(R_X(t_1 - t_2) - m_X^2\right]/\sigma_X^2$ is the correlation coefficient

4.6.1 Important Properties

1. The complete statistical description of a Gaussian RP is obtained by knowing the mean and autocorrelation function

2. Passing a Gaussian RP through an LTI system results in another Gaussian RP

3. For a Gaussian RP WSS and strict sense stationarity are equivalent

4. A sufficient condition for ergodicity of a WSS zero mean Gaussian process is[3]

$$\int_{-\infty}^{\infty} |R_X(\tau)|\, d\tau < \infty$$

5. For a Gaussian process uncorrelatedness and independence are equivalent

[3] E. Wong and B. Hajek, *Stochastic Processes in Engineering Systems*, Springer Verlag, 1985.

4.6.2 Filtered Gaussian Process

- When we input a WSS RP to an LTI system, what can we say about the output?
- In general the answer to this question is not easy
- For the case of a Gaussian RP the answer is easy; it is a Gaussian process!
- Justification, not a rigorous proof, comes by considering the convolution sum

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)\,d\lambda \\ &= \lim_{\Delta\lambda\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta\lambda)h(t-k\Delta\lambda)\Delta\lambda, \end{aligned}$$

  where $h(t)$ is the system impulse response

- From the sum representation we have that the output at time t is a sum of Gaussian random variables, and we know that this is also Gaussian!

4.7 White Processes

A process is termed white, as in white light, if it has a flat or constant power spectral density for all frequencies, i.e.,

$$S_x(f) = \frac{N_0}{2} \text{ for all } f.$$

- Clearly from a practical point-of-view this is impossible since this says the process has infinite power

- No process is truly white, but within the frequency band of interest the spectrum may appear to be white

- The autocorrelation function for a white process is

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

  Note: This also implies that the process is zero mean

- Thermal noise or receiver front-end noise is typically modeled as being a stationary, ergodic, zero mean, white Gaussian process with $N_0 = kT$, where k is Boltzmann's constant (1.38×10^{-23} Joules/Kelvin) and T is temperature in degrees Kelvin

- A quantity communication engineers typically have committed to memory is that at room temperature, i.e., 300° K, $kT = -174$ dBm/Hz

- We frequently combine white and Gaussian together to describe thermal noise as white Gaussian noise (WGN)

- The physics behind this comes from the random motion of charge carriers in a conductor or semiconductor

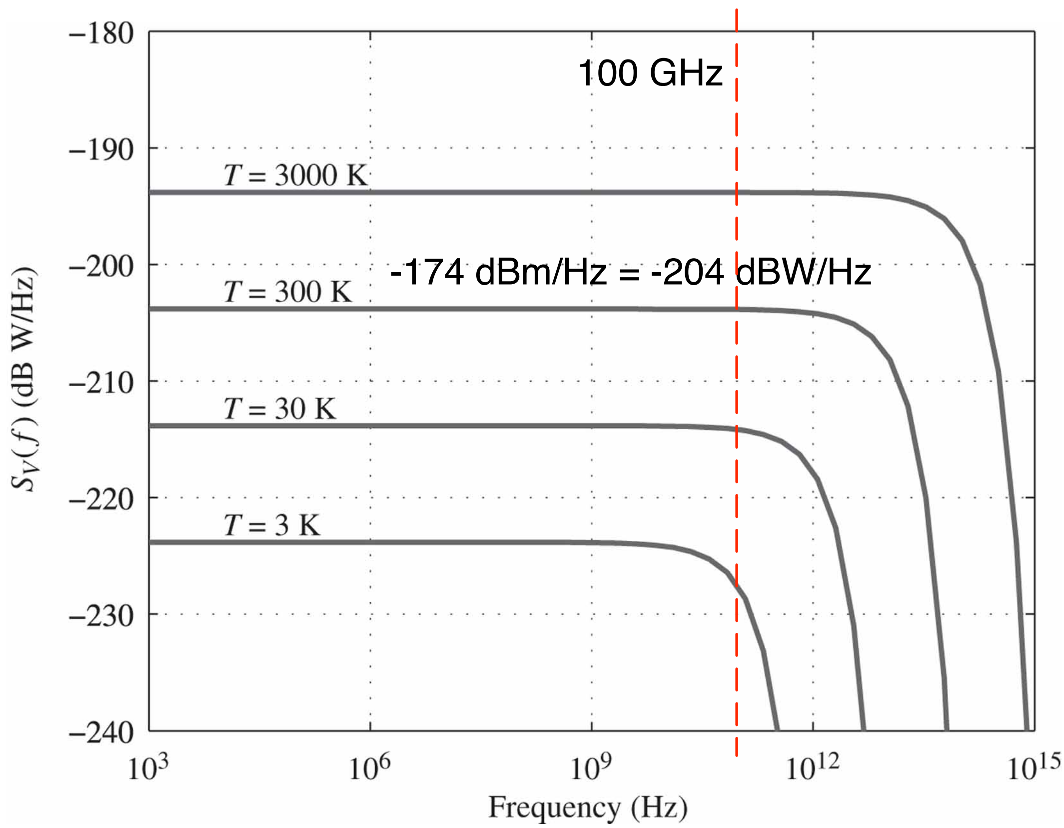
- The noise voltage across the device terminals of say a resistor is a random voltage resulting from the sum of many voltages due to the charge carriers

- From the central limit theorem (law of large numbers) the voltage $V(t)$ tends toward a Gaussian RP

- Nyquist [Rice text] argued that the PSD of thermal noise is

$$S_v(f) = \frac{hf}{\exp\left\{\frac{hf}{kT}\right\} - 1} \text{ W/Hz}$$

where h is Plank's constant, $(6.6261 \times 10^{-34}$ Js)



PSD of thermal noise

- Note how flat the spectrum is, up to 100 GHz and beyond depending upon temperature

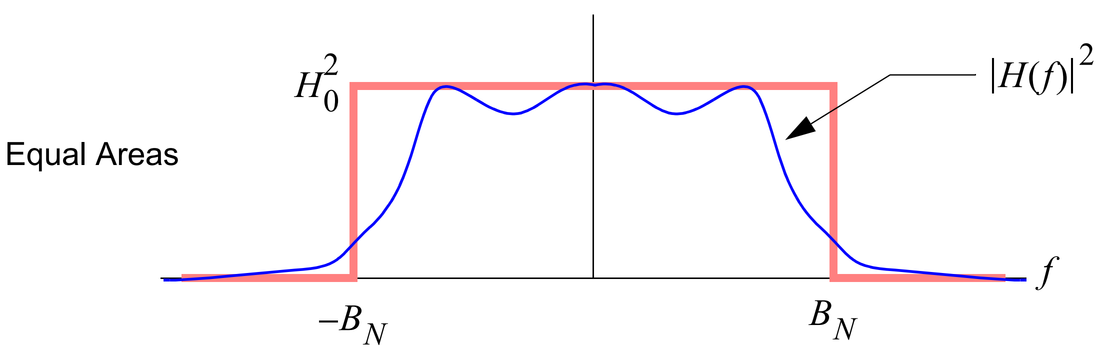
4.8 Noise Equivalent Bandwidth

If we pass a white noise process through a filter with transfer function $H(f)$ we obtain an output power of

$$\sigma_Y^2 = P_Y = \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2 \, df = N_0 \int_0^{\infty} |H(f)|^2 \, df$$

- The noise equivalent bandwidth is the bandwidth of an equivalent ideal filter that passes the same noise power
- Assuming a lowpass filter is being considered with maximum gain of H_0, we wish to equate

$$N_0 H_0^2 B_N = N_0 \int_0^{\infty} |H(f)|^2 \, df$$
$$\Rightarrow B_N = \frac{1}{H_0^2} \int_0^{\infty} |H(f)|^2 \, df$$



Noise equivalent bandwidth concept

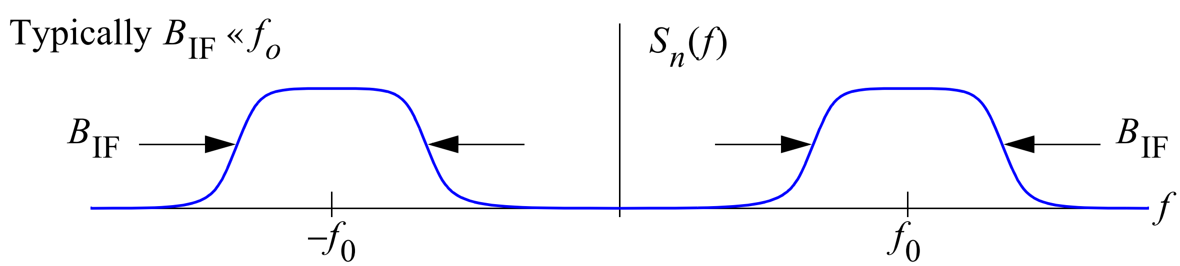
Example 4.12: nth-Order Butterworth Lowpass

- An nth-order Butterworth lowpass filter has noise equivalent bandwidth of the form
$$B_N = \int_0^\infty \frac{1}{1+(f/f_3)^{2n}}\,df = f_3 \int_0^\infty \frac{1}{1+x^{2n}}\,dx$$
$$= \frac{\pi f_3/(2n)}{\sin(\pi/(2n))}$$
for $n = 1, 2, \ldots$, where f_3 is the filter 3dB bandwidth

- Note that as $n \to \infty$, $B_N \to f_3$ (ideal or *brickwall* filter)

4.9 Narrowband Noise

Suppose the spectrum of a zero mean RP, $n(t)$, is zero except in the frequency band B about frequency f_0. Furthermore we typically have $B \ll f_0$.



- An expression for $n(t)$ in terms of lowpass RP's is
$$n(t) = n_c(t)\cos(\omega_0 t + \theta) - n_s(t)\sin(\omega_0 t + \theta)$$
where $n_c(t)$ and $n_s(t)$ are lowpass RP's and θ is an arbitrary phase angle

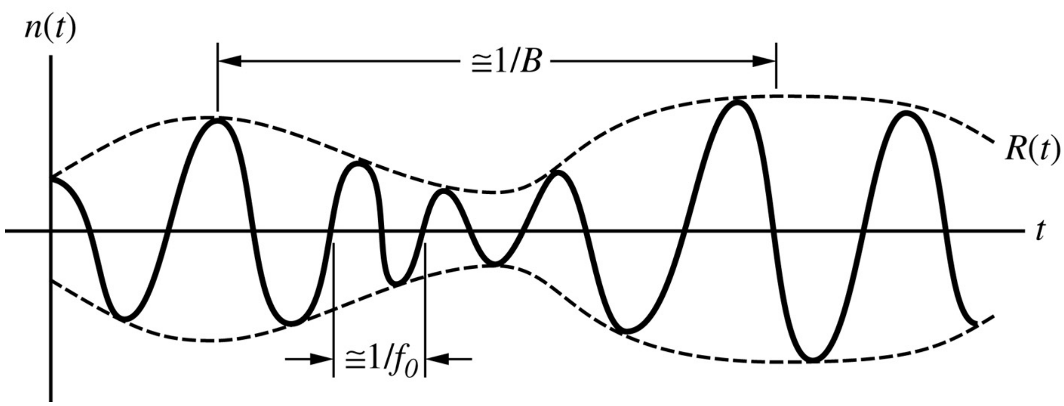
- An equivalent envelope phase representation is

$$n(t) = R(t)\cos\big[2\pi f_0 t + \phi(t) + \theta\big]$$

where

$$R(t) = \sqrt{n_c^2(t) + n_s^2(t)}, \qquad \phi(t) = \tan^{-1}\left(\frac{n_s(t)}{n_c(t)}\right)$$

- The envelope and phase are *slowly varying* with respect to the carrier frequency (note $B = B_{\mathbf{IF}}$)



Narrowband noise and its envelope

- Here we assume $n(t)$ is WSS so $n_c(t)$ and $n_s(t)$ are jointly WSS

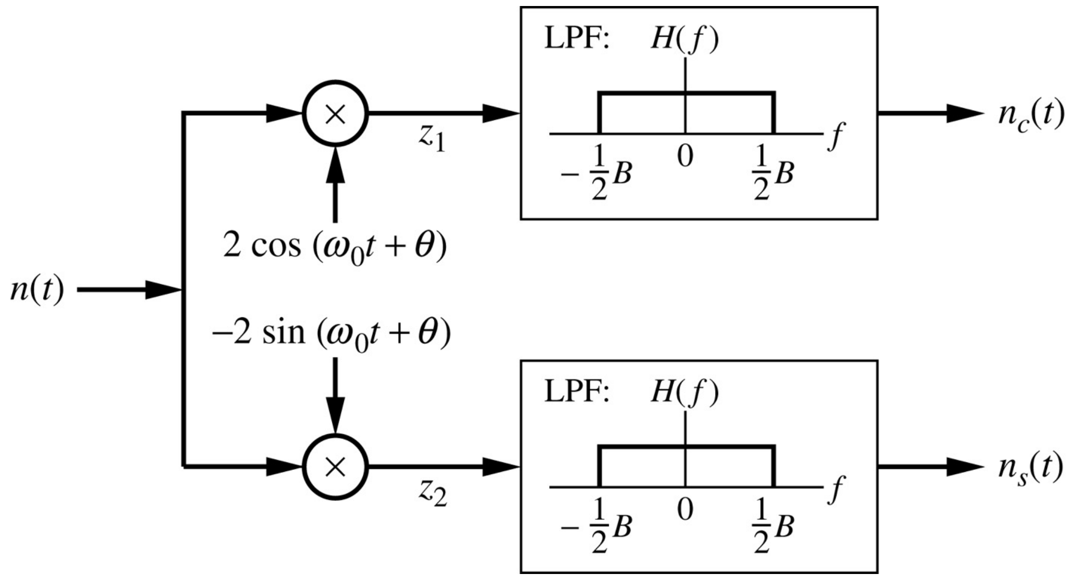
- By direct evaluation $n(t)$ is WSS if

$$R_{n_c}(\tau) = R_{n_c n_c}(\tau) = R_{n_s n_s}(\tau) = R_{n_s}(\tau)$$
$$\text{and } R_{n_c n_s}(\tau) = -R_{n_s n_c}(\tau)$$

- Then,

$$R_n(\tau) = 2\big[R_{n_c}(\tau)\cos(\omega_0\tau) + R_{n_c n_s}(\tau)\sin(\omega_0\tau)\big]$$

- We can view the production of $n_c(t)$ and $n_s(t)$ via:



Conceptual in-phase and quadrature noise generation

- It can be shown that the mean-squared error between $n(t)$ and the narrowband model is zero

- Furthermore, if $n(t)$ is Gaussian, so are $n_c(t)$ and $n_s(t)$

Useful Properties

- **Means**

$$m_n(t) = m_{n_c}(t) = m_{n_s}(t) = 0$$

- **Variances**

$$\sigma_n^2 = \sigma_{n_c}^2 = \sigma_{n_s}^2 \stackrel{\Delta}{=} N$$

- **PSDs**

  Analysis:

$$\begin{aligned} S_n(f) = &\left[S_{n_c}(f - f_0) + S_{n_c}(f + f_0)\right] \\ &- j\left[S_{n_c n_s}(f - f_0) - S_{n_c n_s}(f + f_0)\right] \end{aligned}$$

Synthesis:

$$S_{n_c}(f) = S_{n_s}(f) = \mathrm{Lp}\big[S_n(f - f_0) + S_n(f + f_0)\big]$$

where Lp[ ] denotes the lowpass part of the quantity in brackets

- **Cross-PSD**

$$S_{n_c n_s}(f) = j\,\mathrm{Lp}\big[S_n(f - f_0) - S_n(f + f_0)\big]$$

- Note that if the cross-PSD is zero, then it follows that $R_{n_c n_s}(\tau) \equiv 0$ for all values of τ, and hence that $n_c(t)$ and $n_s(t)$ are mutually uncorrelated processes!

- The above results hold when $n(t)$ is symmetrical about f_0

- Given that $n(t)$ is also Gaussian, the symmetry condition makes the joint pdf between $n_c(t)$ and $n_s(t)$ for any delay of the form

$$f_{n_c n_s}(n_c, t + \tau; n_s, t) = \frac{1}{2\pi N} e^{-(n_c^2 + n_s^2)/2N}$$

- When $S_n(f)$ is not symmetrical about f_0, then the above pdf result holds only for $\tau = 0$

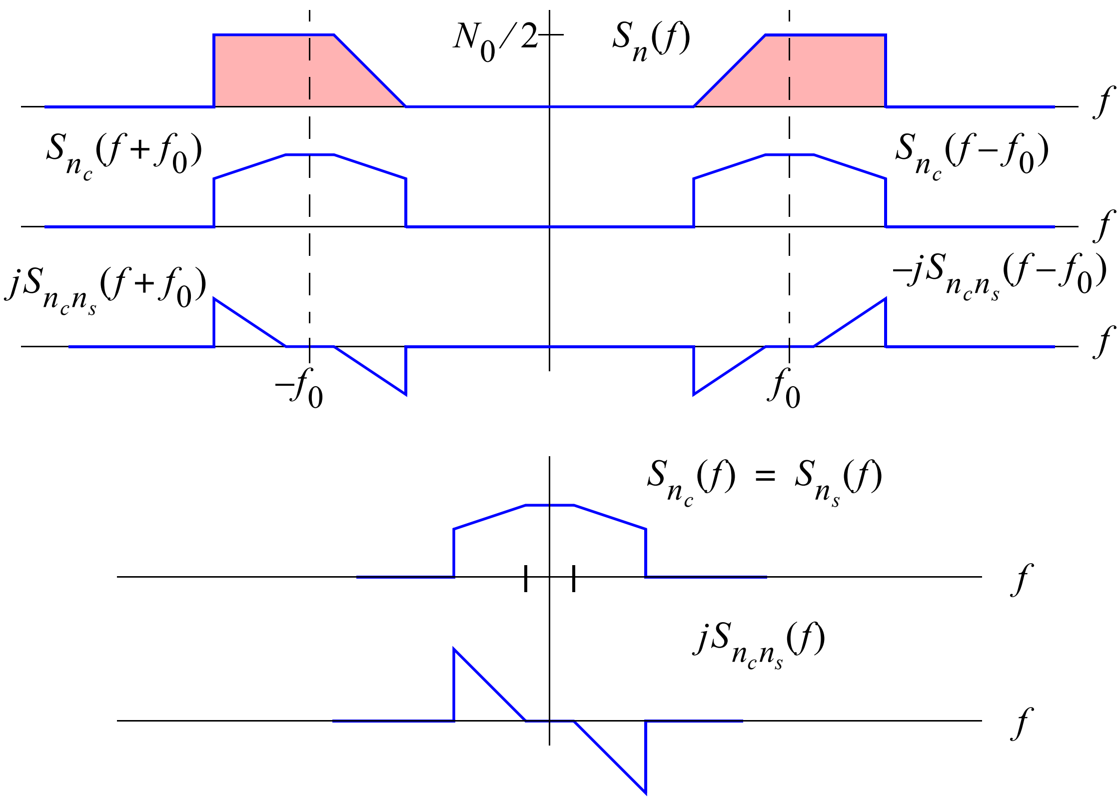
- Considering the narrowband process envelope and phase functions, the joint pdf is

$$f_{R,\Phi}(r, \phi) = \frac{r}{2\pi N} e^{-r^2/2N},\; r > 0 \text{ and } |\phi| \leq 0$$

(Rayleigh envelope and uniform phase)

- Proof: Consult Z&T p. 333–336

Example 4.13: **General Case Pictorial**

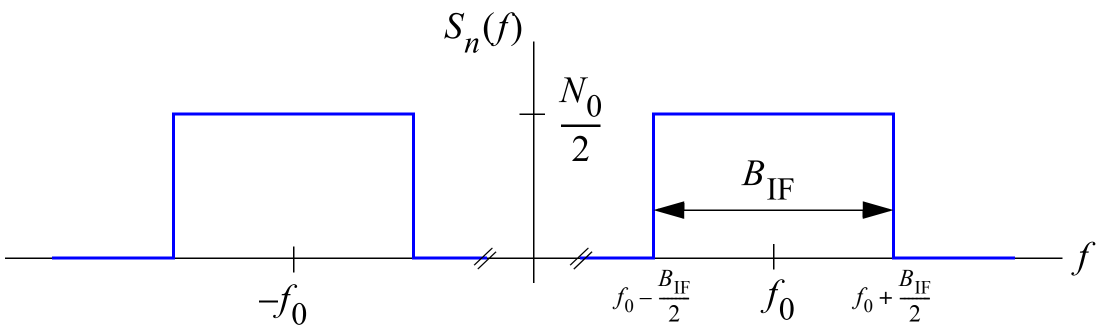


General asymmetric $n(t)$ in pictures

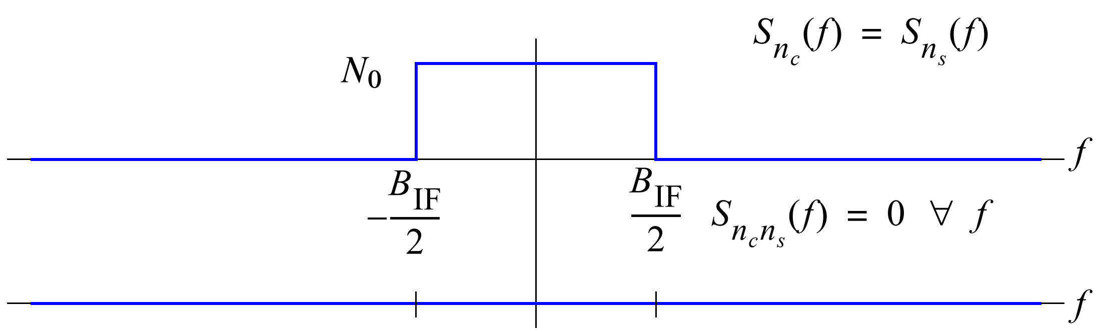
- Without symmetry the nice properties do not hold
- It is still true that $n_c(t+\tau)$ and $n_s(t)$ will be uncorrelated at $\tau = 0$

Example 4.14: Symmetric Gaussian RP

- Let $n(t)$ be a Gaussian RP with spectrum symmetrical about f_0, with flat spectral density over $B = B_{\text{IF}}$ of $N_0/2$



- It follows that



- Note that

$$P_n = \sigma_n^2 = R_n(0) = \int_{-\infty}^{\infty} S_n(f)\, df$$
$$= \frac{N_0}{2} \cdot 2B_{\text{IF}} = N_0 B_{\text{IF}}$$

and

$$P_{n_c} = P_{n_s} = R_{n_c}(0) = \int_{-\infty}^{\infty} S_{n_c}(f)\, df = N_0 B_{\text{IF}}$$

also, since $S_{n_c n_s}(f) = 0$ for all f, we have

$$R_{n_c n_s}(\tau) = \mathcal{F}^{-1}\{S_{n_c n_s}(f)\} = 0$$

for all τ, which implies that $n_c(t)$ and $n_s(t)$ are uncorrelated for **any** delay τ

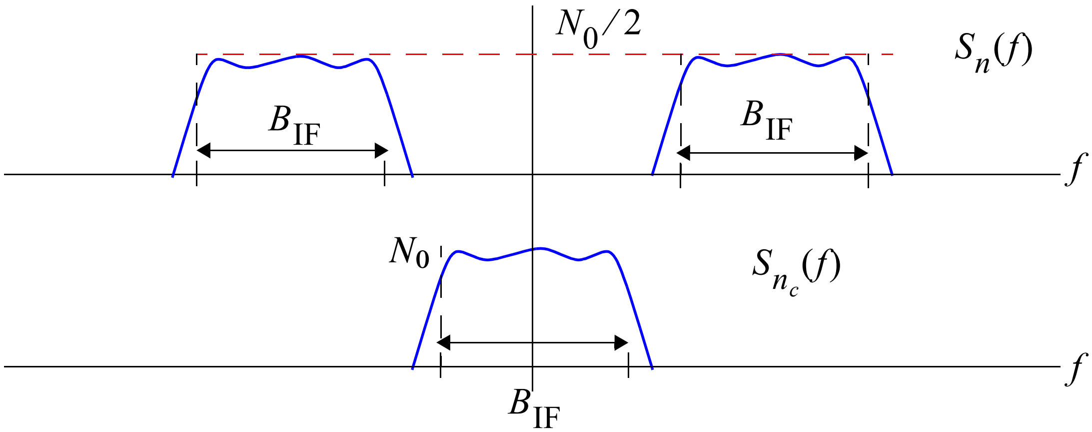
- Since $n_c(t)$ and $n_s(t)$ are also Gaussian, they are also independent

- The PDFs are

$$\begin{aligned} f_{n_c}(n_c) &= \frac{1}{\sqrt{2\pi(N_0/2)B_{\text{IF}}}} e^{-n_c^2/(N_0 B_{\text{IF}})} \\ f_{n_s}(n_s) &= \frac{1}{\sqrt{2\pi(N_0/2)B_{\text{IF}}}} e^{-n_s^2/(N_0 B_{\text{IF}})} \\ f_{n_c n_s}(n_c, n_s) &= \frac{1}{2\pi(N_0/2)B_{\text{IF}}} e^{-(n_c^2+n_s^2)/(N_0 B_{\text{IF}})} \end{aligned}$$

4.9.1 Equivalent Noise Bandwidth for the $n(t)$ Process

- Independent of $n(t)$ being symmetrical about f_0, it may also be nonflat, e.g.,



Non-flat narrowband noise

- An equivalent rectangular noise power bandwidth is obtained by setting

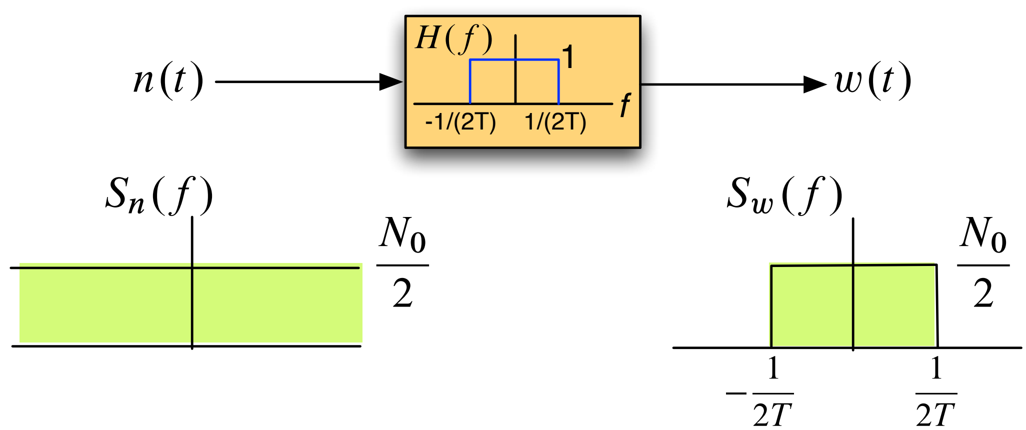
$$\int_{-\infty}^{\infty} S_n(f)\, df = N_0 B_{\mathrm{IF}}$$

or

$$B_{\mathrm{IF}} = \frac{1}{N_0}\int_{-\infty}^{\infty} S_n(f)\, df = \frac{2}{N_0}\int_{0}^{\infty} S_n(f)\, df$$
$$= \int_{0}^{\infty} \frac{S_n(f)}{S_{n_c}(0)}\, df \text{ Hz}$$

4.9.2 Discrete-Time Domain Considerations

- We have been considering continuous-time RPs up to this point
- In the discrete-time domain $X(t) \to X[n]$ and the ensemble of sample functions is replaced by an ensemble of random sequences
- All of the concepts still apply
- What about white Gaussian noise? It is not bandlimited and hence violates the sampling theorem
- Suppose the WGN is lowpass filtered to the Nyquist frequency, $1/(2T)$



Bandlimited WGN ready for sampling at rate $1/T$

- Inverse Fourier transforming the spectrum $S_w(f)$ results in

$$R_w(\tau) = \frac{N_0}{2T}\text{sinc}(\tau/T)$$

- When $w(t)$ sampled at rate $f_s = 1/T$ we obtain $w[n]$, which has autocorrelation sequence

$$R_{ww}[k] = E\big[w[n+k]w[n]\big] = R_w(kT) = \frac{N_0}{2T}\delta[k]$$

- Note that with this filtering and sampling arrangement, the noise process although bandlimited and non-white, when sampled at the proper sampling instants appears to be white

4.9.3 Ricean PDF

- In a fading channel environment an unmodulated carrier can be represented as the sum of a randomly phase sinusoid plus bandlimited Gaussian noise, i.e.,

$$z(t) = \underbrace{A\cos(2\pi f_0 t + \theta)}_{\text{specular}} + \underbrace{n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)}_{\text{diffuse}}$$

- In this model we assume that A is a constant and θ is $\mathcal{U}(0, 2\pi)$

- The first term is known as the *specular* component and the second two make up the *diffuse* component

- The specular component results from reception of a direct-ray of a signal in a multipath environment and the diffuse component results from multiple independent reflections (think mobile radio where you can see the transmit antenna, but multiple reflectors are still present)

- The in-phase and quadrature terms of the diffuse component are Gaussian RPs

- The Ricean pdf is obtained by first writing

$$A\cos(\omega_0 t + \theta) = A\cos\theta\cos(2\pi\omega_0 t) - A\sin\theta\sin(2\pi\omega_0 t)$$

- Combining cos and sin terms in $z(t)$ we have

$$\begin{aligned} z(t) &= \left[A\cos\theta + n_c(t)\right]\cos(2\pi f_0 t) \\ &\quad - \left[A\sin\theta + n_s(t)\right]\sin(2\pi f_0 t) \\ &= X(t)\cos(2\pi f_0 t) - Y(t)\sin(2\pi f_0 t) \end{aligned}$$

  where

$$X(t) = A\cos\theta + n_c(t) \quad \text{and} \quad Y(t) = A\sin\theta + n_s(t)$$

- We desire the pdf of the envelope, $R(t) = \sqrt{X^2(t) + Y^2(t)}$

- We use the rectangular to polar conversion transformation of two RVs into 2 RVs described in the previous chapter

- We obtain the joint pdf by this method, but can then integrate out the unwanted RV ϕ to obtain the pdf on R alone

- It can be shown that

$$f_R(r) = \frac{r}{\sigma_2}\exp\left[-\left(r^2 + A^2\right)/(2\sigma^2)\right] I_0\left(\frac{Ar}{\sigma^2}\right),\ r \geq 0$$

  where σ^2 is the variance of the Gaussian RP and $I_0(\,)$ is

$$I_0(u) = \frac{1}{2\pi}\int_0^{2\pi} \exp(u\cos\alpha)\,d\alpha$$

  is the the modified Bessel function of order zero (available in MATLAB)

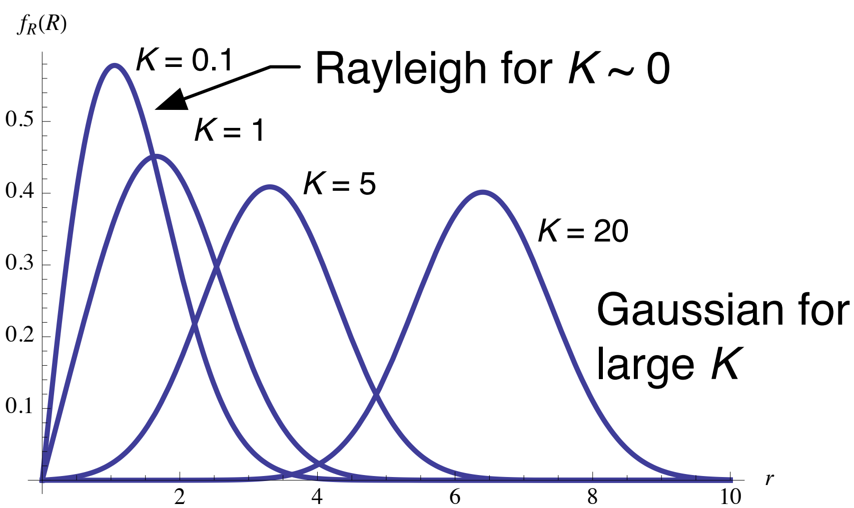
- In practice we parameterize the pdf in terms of the *Ricean K-factor* defined as $K = A^2/(2\sigma^2)$

- For large K the specular component dominates and the pdf is Gaussian
- For $K = 0$ the diffuse component dominates and we have a Rayleigh PDF

```
fRICE[r_, K_, σ_] := r/σ^2 Exp[-(r^2/(2 σ^2) + K)] BesselI[0, Sqrt[2 K] r/σ] UnitStep[r];

K = {0.1, 1, 5, 20};

Plot[Table[fRICE[r, K[[n]], 1], {n, 1, 4}], {r, 0, 10},
 PlotStyle → Thick, PlotRange → All, AxesLabel → {"r", "fR(R)"}]
```



Ricean pdf versus K with $\sigma^2 = 1$

- **Moments**: It can be shown that

$$E[R^2] = E[X^2] + E[Y^2] = 2\sigma^2(1 + K)$$