Sample Moments for Scalar and Vector Random Variables

The sample mean, sample variance, and sample covariance are simple measurement techniques to characterize random quantities. Block forms and recursive forms of these algorithms are also available.

Sample Mean

For random number sequence $x(n)$, a sample drawn from random variable $\mathbf{x}$, the sample mean is

$$m_x = E\{\mathbf{x}\} \cong \frac{1}{N}\sum_{k=1}^{N} x(k) = \hat{m}_x \tag{1}$$

A recursive formula for computing $\hat{m}_x$ is

$$\hat{m}_x(N) = \hat{m}_x(N-1) + \frac{1}{N}[x(N) - \hat{m}_x(N-1)] \tag{2}$$

where $\hat{m}_x(0) = 0$.

Sample Variance

To estimate the quantity $\sigma_x^2 = E\{[\mathbf{x} - m_x]^2\}$ use

$$\sigma_x^2 \cong \frac{1}{N}\sum_{k=1}^{N} [x(k) - \hat{m}_x]^2 = \hat{\sigma}_x^2 \tag{3}$$

Note this is a biased version of the variance estimator presented in the course notes.

A recursive formula for calculating $\hat{\sigma}_x^2$ is

$$\hat{\sigma}_x^2(N) = \frac{N-1}{N}\hat{\sigma}_x^2(N-1) + \frac{N-1}{N^2}[x(N) - \hat{m}_x(N-1)]^2 \tag{4}$$

where $\hat{\sigma}_x^2(0) = 0$ and $\hat{\sigma}_x^2(1) = 0$. With some work this can be converted to the unbiased estimator. Basically we need to replace N with $N-1$.

Sample Mean Vector

Consider the random vector $\mathbf{X} = \left[x(1)\ x(2)\ \ldots\ x(n)\right]^t$ which has corresponding mean vector $\mathbf{m}_x = \left[m_{x_1}\ m_{x_2}\ \ldots\ m_{x_n}\right]^t$. The sample mean vector is given by

$$\hat{\mathbf{m}}_x \cong \frac{1}{N}\sum_{k=1}^{N} \mathbf{X}(k) \tag{5}$$

The recursive formula presented earlier for the scalar rv case can also be extended to the vector case.

Sample Covariance Matrix

The random vector $\underline{X}$ defined above has covariance matrix given by

$$\begin{aligned} \mathbf{C} &= E\{[\mathbf{X} - \mathbf{m}_x][\mathbf{X} - \mathbf{m}_x]^t\} \\ &= \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix} \end{aligned} \tag{6}$$

An estimator (biased) for $\underline{C}$ is

$$\mathbf{C} \cong \frac{1}{N}\sum_{k=1}^{N} \mathbf{X}(k)\mathbf{X}^t(k) - \hat{\mathbf{m}}_x\hat{\mathbf{m}}_x^t = \hat{\mathbf{C}} \tag{7}$$

A recursive formula for $\hat{\mathbf{C}}$ is

$$\begin{aligned} \hat{\mathbf{C}}(k) &= \frac{1}{N}[(N-1)(\hat{\mathbf{C}}(N-1) + \hat{\mathbf{m}}_x(N-1)) + \mathbf{X}(N)\mathbf{X}^t(N)] \\ &-\frac{1}{N^2}[(N-1)\hat{\mathbf{m}}_x(N-1) + \mathbf{X}(N)][(N-1)\hat{\mathbf{m}}_x(N-1) + \mathbf{X}(N)]^t \end{aligned} \tag{8}$$

where $\hat{\mathbf{C}}(0) = \underline{0}$ and $\hat{\mathbf{C}}(1) = \underline{0}$. Note this biased estimator can be converted to an unbiased estimator if desired.

Sample Autocorrelation Function

If we view the random number sequence $x(n)$ as sample function from a wide-sense stationary discrete-time random process, then we can write

$$R_{xx}(k) \cong \frac{1}{N}\sum_{n=0}^{N+|k|-1} x(n+k)x(n) = \hat{R}_N(k) \tag{9}$$

In the above expression it is assumed that the available data is the real sequence $x(0)$,

$x(1), \ldots, x(N-1)$, thus $\hat{R}_N(k) = 0$ for $|k| \geq N$. Note that this sequence is also biased since

$$E\{\hat{R}_N(k)\} = \left(1 - \frac{|k|}{N}\right) R_{xx}(k) \tag{10}$$

but $\hat{R}_N(k)$ is asymptotically unbiased since $\lim_{n \to \infty}(1 - |k|/N) = 1$. The bias is not that significant since $N \gg |k|$ is needed for the estimate variance to be small.