An Upper Bound on the Cardinality of a Minimum Feedback Vertex Set for Directed Graphs

Abstract

We derive an upper bound on the cardinality of a minimum feedback vertex set for arbitrary directed graphs, and provide example graphs which achieve this bound.

Let $G = (V, E)$ be a directed, not necessarily connected graph. A subset of vertices $M \subseteq V$ is called a *minimum feedback vertex set* if the subgraph induced by the removal of M from G contains no directed cycles and M is the smallest such set of vertices. The problem of finding a minimum feedback vertex set in an arbitrary graph is known to be NP-hard even in sparse graphs [1], [2] and its decision version was one of Karp's original 21 NP-complete problems [3]. Nonetheless, identifying minimum feedback vertex sets in directed graphs has applications in a number of disparate domains, including deadlock recovery in operating systems and computer engineering [4], and determining the inefficiency of Nash equilibria in game theory [5].

Recent years have seen considerable attention focused on deriving bounds on the cardinality of the minimum feedback vertex set for undirected graphs. For instance see bounds for planar graphs [6], hypercubes [7], shuffles [8], and further conjectures [9]. However, there seems to be a lack of similar results for the valuable case of directed graphs. Accordingly, this note presents a simple upper bound for this quantity for directed graphs as a function of the number of nodes and edges. We present graphs that achieve this bound. While tight in general, this bound can almost certainly be improved on specific classes of graphs, and we invite improvements from the mathematical community.

I. Preliminaries

We present the following definitions to ensure our results are clear. A directed graph $G = (V, E)$ (or *digraph*) is specified by vertex set V and edge set E and we assume that G is simple (i.e., we forbid repeated edges or self-loops), but we do not require that G is connected. A *directed cycle* is a sequence of distinct edges $\{e_1, e_2, \ldots, e_m\}$ and a corresponding sequence of distinct vertices $\{v_1, v_2, \ldots, v_m\}$ such that for all $i < m$, $e_i = (v_i, v_{i+1})$ and $e_m = (v_m, v_1)$. Note that a directed cycle can have length 2; i.e., the graph $(\{v_1, v_2\}, \{(v_1, v_2), (v_2, v_1)\})$ has a directed cycle. A digraph G is called *acyclic* if if it contains no directed cycles. A set of vertices $M \subseteq V$ is called a *feedback vertex set* if the graph $(V \setminus M, E \setminus \{(i, j) \in E \mid i \in M \text{ or } j \in M\})$ is acyclic. The set M is called a *minimum feedback vertex set* if it is a feedback vertex set of minimum cardinality. We write $\alpha(G)$ to denote the cardinality of a minimum feedback vertex set for digraph G.

II. Contribution

Our main result gives a tight upper bound for all (not necessarily connected) digraphs as a function of the number of edges.

Theorem 1. *Let $G = (V, E)$ be a digraph. Then*

$$\alpha(G) \leq \min \left\{ |V| - 1, \left\lfloor \frac{|E|}{2} \right\rfloor \right\}. \tag{1}$$

There exist graphs which achieve this bound.

Proof. First we show the edge-based upper bound. Let G be a directed graph, and let $M(G)$ be a minimum feedback vertex set of G. We wish to show that

$$2\alpha(G) \leq |E|. \tag{2}$$

Enumerate the m vertices in $M(G) := \{u_1, u_2, \ldots, u_m\}$. Define the sequence $\{G_i\}_{i=0}^m$ as follows: $G_0 = G$, and G_i is the digraph that results from the removal of u_i and all incident edges from G_{i-1}. By the definition of $M(G)$, each vertex u_i is contained in at least one cycle in graph G_{i-1}. Since each u_i is contained in a cycle in G_{i-1}, it follows that at least 2 edges are incident on u_i in G_{i-1}. Since the sequence of graphs satisfies $E(G_i) \subset E(G_{i-1})$, this means that for all $i = 1, \ldots, m$, at least 2 unique edges are incident on vertex u_i which are *not* incident on u_j for any $j < i$. Thus, writing $E(M(G))$ to denote the set of edges incident on $M(G)$, we have that

$$2\alpha(G) \leq |E(M(G))| \leq |E|. \tag{3}$$

To show the edge-based lower bound, we construct a family of graphs which satisfy (1) with equality for all k, where $|E| = k$. Let $|V| = 2\lfloor k/2 \rfloor$, and let $E = \{(2i, 2i-1) \cup (2i-1, 2i) \mid i \in \{1, \ldots, |V|/2\}\}$ (see Figure 1). If $|E|$ is odd, define the single remaining edge between any feasible pair of vertices (e.g., $(1, 3)$). Since this graph contains exactly $|V|/2$ disjoint length-2 directed cycles, the cardinality of any minimum feedback vertex set is exactly $|V|/2$ (e.g., the set of odd-numbered vertices is such a set).

The vertex-based upper bound is trivial, since if $|V| - 1$ vertices and associated edges are removed from any graph, the resulting graph is a single vertex and thus acyclic. The vertex-based lower bound is achieved by a complete digraph. $\square$

References

[1] F. V. Fomin, S. Gaspers, and A. V. Pyatkin, "Finding a minimum feedback vertex set in time $O(1.7548ˆn)$," in *International Workshop on Parameterized and Exact Computation*, 2006, pp. 184–191. [Online]. Available: https://link.springer.com/content/pdf/10.1007/11847250_17.pdf

[2] G. Borradaile, H. Le, and B. Zheng, "Engineering a PTAS for Minimum Feedback Vertex Set in Planar Graphs," in *International Symposium on Experimental Algorithms*, 2019, pp. 98–113. [Online]. Available: http://link.springer.com/10.1007/978-3-030-34029-2_7

[3] R. M. Karp, "Reducibility among Combinatorial Problems," in *Complexity of Computer Computations*. Boston, MA: Springer US, 1972, pp. 85–103. [Online]. Available: http://link.springer.com/10.1007/978-1-4684-2001-2_9

[4] Hen-Ming Lin and Jing-Yang Jou, "On computing the minimum feedback vertex set of a directed graph by contraction operations," *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, vol. 19, no. 3, pp. 295–307, mar 2000. [Online]. Available: http://ieeexplore.ieee.org/document/833199/

[5] P. N. Brown and J. R. Marden, "On the feasibility of local utility redesign for multiagent optimization," in *18th European Control Conference*, 2019, pp. 3396–3401. [Online]. Available: https://ieeexplore.ieee.org/document/8795902

[6] T. Kelly and C.-H. Liu, "Minimum size of feedback vertex sets of planar graphs of girth at least five," *European Journal of Combinatorics*, vol. 61, pp. 138–150, mar 2017. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0195669816301032

[7] F. R. Madelaine and I. A. Stewart, "Improved upper and lower bounds on the feedback vertex numbers of grids and butterflies," *Discrete Mathematics*, vol. 308, no. 18, pp. 4144–4164, sep 2008. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0012365X07006267

[8] R. Královič and P. Ružička, "Minimum feedback vertex sets in shuffle-based interconnection networks," *Information Processing Letters*, vol. 86, no. 4, pp. 191–196, may 2003. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0020019002005045

[9] Ł. Kowalik, B. Lužar, and R. Škrekovski, "An improved bound on the largest induced forests for triangle-free planar graphs," *Discrete Mathematics and Theoretical Computer Science*, vol. 12, no. 1, pp. 87–100, 2010. [Online]. Available: https://dmtcs.episciences.org/487

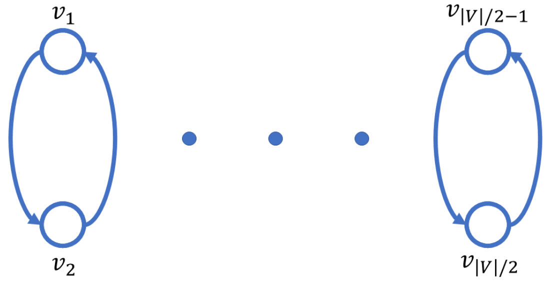

Fig. 1. Graph with $\alpha(G) = \lfloor |E|/2 \rfloor$, achieving the bound given in Theorem 1 in (1) (Graphe avec $\alpha(G) = \lfloor |E|/2 \rfloor$, qui atteignent le majorant de Theorem 1).