ENERGY MODEL WITH DISTANCE DISTRIBUTION

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I. Opportunistic Griding

Data traffic in sensor networks follows many-to-one pattern, ussally the area close to the sink is always crowded if uniform grid is used [1]. *Opportunistic griding* is therefore advantageous in smoothing energy distribution. Grids close to the sink, which have heavier traffic load, will have a smaller size compared with those are farther away, such that a shorter transmission range is used. Sensors that are far away have less traffic, but have to transmit over longer distance. This method will intuitively balance out the uneven traffic distribution due to geometric location, and thus make network lifetime much longer.

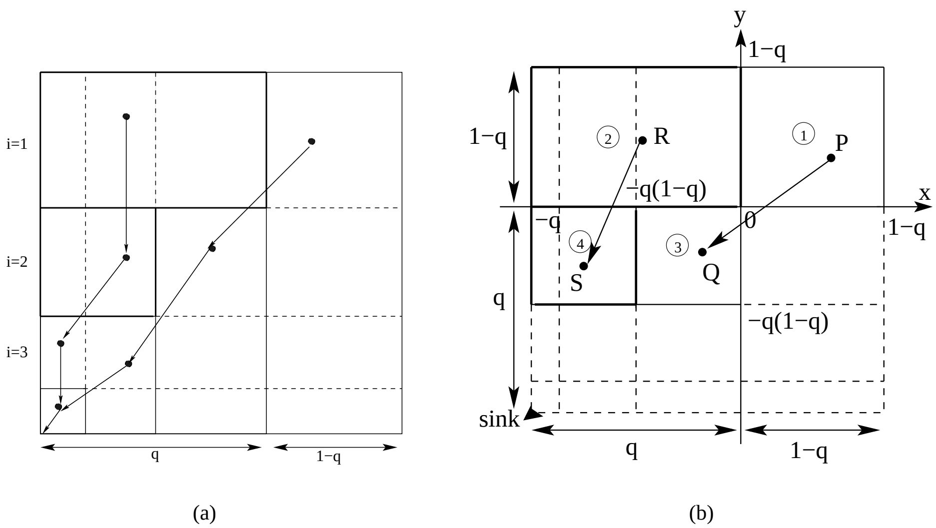


Fig. 1. Opportunistic Griding with Size Ratio q.

In Fig. 1 (a), the area of the entire sensing field is A^2. Sink node locates at the lower left corner, and the field is recursively divided with size ratio q. Notice that if the sink node is located at an arbitrary position, this network division can be repeated in four different quadrants with sink node as the origin. Without loss of generality, we normalize the $A \times A$ sensing field to a unit square as in Fig. 1 (b). With a grid-based clustering scheme, there are three possible cases of transceiver locations for a wireless transmission: between a cluster node and the grid head within the same grid; between two grid heads of neighbor grids, where the grids can be diagonal (PQ in grid 1 and 3) or parallel (RS in grid 2 and 4) to each other. In the following sections, we first deal with distance distribution with uniform size, i.e., uniform griding, then derive the more complicated cases with the size ratio of q.

II. Distance Distributions

As long as the coordinate distributions of two nodes are obtained, we can get the distance distribution according to its definition, i.e., $P(D \leq d)$ where $D = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$. Let $V = X_1 - X_2$ (or $Y_1 - Y_2$), $S = V^2$, $Z = S_X + S_Y$ and $D = \sqrt{Z}$, our goal is to obtain $f_D(d)$, i.e., the distance probability density function of the two cases described above.

1) Difference Distribution: With $V = X_1 - X_2$, the cumulative distribution function of V is $F_V(v) = P(X_1 - X_2 \leq v) = \int_{x_1-v}^{\infty}\left[\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)dx_1\right]dx_2 = \int_{-\infty}^{v}\left[\int_{-\infty}^{\infty} f_{X_1,X_2}(x, x-v)dx\right]dv$, where $f_{X_1,X_2}(x_1,x_2)$ is the joint probability density function of X_1 and X_2. Since X_1 and X_2 are independent random variables, $F_V(v) = \int_{-\infty}^{v}\left[\int_{-\infty}^{\infty} f_{X_1}(x)f_{X_2}(x-v)dx\right]dv$. Let the probability density function be $f_V(v)$, we have

$$f_V(v) = \int_{-\infty}^{\infty} f_{X_1}(x)f_{X_2}(x-v)dx. \tag{1}$$

2) Square Distribution: With $S = V^2$, $F_S(s) = P(V^2 \leq s) = P(-\sqrt{s} \leq V \leq \sqrt{s}) = \int_{-\sqrt{s}}^{\sqrt{s}} f_V(v)dv = F_V(\sqrt{s}) - F_V(-\sqrt{s})$, then

$$f_S(s) = F'_S(s) = \frac{f_V(\sqrt{s}) + f_V(-\sqrt{s})}{2\sqrt{s}}. \tag{2}$$

3) Sum Distribution: With $Z = S_X + S_Y$, following the same approach in *Difference Distribution*, $F_Z(z) = P(S_X + S_Y \leq z) = \int_{-\infty}^{z}\left[\int_{-\infty}^{\infty} f_{S_X,S_Y}(x, z-x)dx\right]dz$. The following convolution is used to obtain $f_Z(z)$:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{S_X}(x)f_{S_Y}(z-x)dx. \tag{3}$$

4) Square-Root Distribution: Finally let $D = \sqrt{Z}$, the distance distribution $f_D(d)$ can be derived by

$$f_D(d) = F'_Z(d^2) = 2df_Z(d^2). \tag{4}$$

III. Distance Distribution With Uniform Size

In the network topology described in Fig. 1, data can be transmitted in three cases: between nodes inside the same grid, between nodes in diagonally adjacent grids, and nodes in vertically or horizontally adjacent grids. In most literature, average distance is used, instead of distance distribution, which is more accurate according to simulation results in the next section. We derive the distance distribution in these three cases.

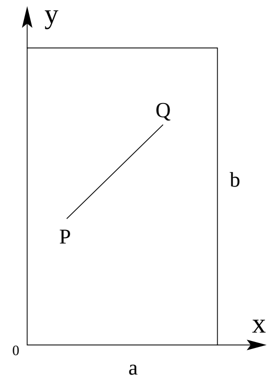


Fig. 2. Random Points Within a Grid.

A. Two Random Points Within a Grid

Consider a rectangle with the length of the sides a and b, and suppose $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two random points that uniformly distributed in this rectangle area, as in Fig. 2. Then the cordinates of P and Q are random variables that follow uniform distribution: $x_1, x_2 \sim U[0, a]$ and $y_1, y_2 \sim U[0, b]$. Let $v_1 = x_1 - x_2$, $v_2 = y_1 - y_2$. The probability density function can be easily computed by introducing *delta function* $\delta(x)$ [2] and *Heaviside step function* $H(x)$ [3]:

$$f_V(v) = \int_0^a \int_0^a \delta\left(\frac{x - y - v}{a}\right) dxdy \tag{5}$$

Therefore,

$$\begin{aligned} f_{V_1}(v) &= \tfrac{(v+a)H(v+a)+(v-a)H(v-a)-2vH(v)}{a^2} \quad and \\ f_{V_2}(v) &= \tfrac{(v+b)H(v+b)+(v-b)H(v-b)-2vH(v)}{b^2} \end{aligned} \tag{6}$$

In simpified form,

$$f_{V_1}(v) = \tfrac{v+a-2vH(v)}{a^2} \quad and$$
$$f_{V_2}(v) = \tfrac{v+b-2vH(v)}{b^2}$$

$s_1 = v_1^2 = (x_1 - x_2)^2$, $s_2 = v_2^2 = (y_1 - y_2)^2$ has pdf:

$$f_{S_1}(s) = \tfrac{H(\sqrt{s}+a)-H(\sqrt{s}-a)}{a\sqrt{s}} + \tfrac{H(\sqrt{s}+a)+H(\sqrt{s}-a)-2H(\sqrt{s})}{a^2} \quad and$$
$$f_{S_2}(s) = \tfrac{H(\sqrt{s}+b)-H(\sqrt{s}-b)}{b\sqrt{s}} + \tfrac{H(\sqrt{s}+b)+H(\sqrt{s}-b)-2H(\sqrt{s})}{b^2}$$

In simpified form,

$$f_{S_1}(s) = \frac{1}{a\sqrt{s}} + \frac{1-2H(\sqrt{s})}{a^2} \quad and$$
$$f_{S_2}(s) = \frac{1}{b\sqrt{s}} + \frac{1-2H(\sqrt{s})}{b^2}$$

Computing the density function of $z = s_1 + s_2$ is actually doing this convolution (to get the pdf of $d = \sqrt{z}$ is therefore simply $f(d) = 2df_Z(d^2)$):

$$f_Z(z) = \int_{-\infty}^{\infty} f_{S_1}(x) f_{S_2}(z - x)dx \tag{7}$$

Mathai [4] has done impresive work in geometrical probability. With slight modification of his equation $(2.4.9)$, we can get the pdf in terms of node distance, i.e., $d = \sqrt{z}$:

$$f(d) = \frac{4d}{a^2b^2} \begin{cases} \frac{ab\pi}{2} - (a+b)d + \frac{d^2}{2} & 0 \le d \le min\{a,b\} \\ absin^{-1}(\frac{min\{a,b\}}{d}) - \frac{min\{a,b\}^2}{2} - max\{a,b\}d & \\ +max\{a,b\}\sqrt{d^2 - min\{a,b\}^2} & min\{a,b\} \le d \le max\{a,b\} \\ ab\left[sin^{-1}(\frac{min\{a,b\}}{d}) - sin^{-1}\sqrt{1 - \frac{max\{a,b\}^2}{d^2}}\right] & \\ +min\{a,b\}\sqrt{d^2 - max\{a,b\}^2} & \\ +max\{a,b\}\sqrt{d^2 - min\{a,b\}^2} & max\{a,b\} \le d \le \sqrt{a^2+b^2} \\ -\frac{a^2+b^2}{2} - \frac{d^2}{2} & \\ 0 & otherwise \end{cases} \tag{8}$$

B. Two Random Points in Diagonal Grids (Uniform Size)

We first consider simple cases where grid size is the same. In Fig. 3, $x_1, y_1 \sim U[0,s]$, $x_2, y_2 \sim U[-s,0]$, therefore both $v_1 = x_1 - x_2$ and $v_2 = y_1 - y_2$ have the same distribution (simpified form):

$$f_V(v) = \frac{v - 2(v-s)H(v-s)}{s^2} \tag{9}$$

Let $a = v^2$, therefore,

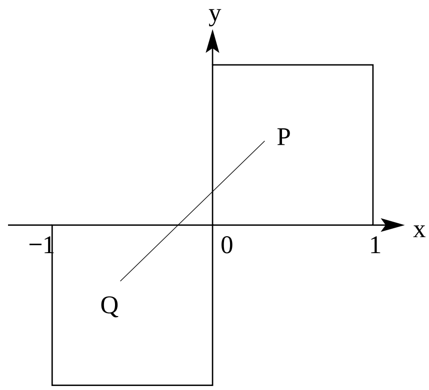


Fig. 3. Random Points in Diagonal Grids (Uniform Size).

$$f_A(a) = \frac{2H(s-\sqrt{a})-1}{2s^2} + \frac{H(\sqrt{a}-s)}{s\sqrt{a}} \tag{10}$$

The same as equation (7), doing convolution $\int_{-\infty}^{-\infty} f_A(x)f_A(z-x)dx$, then apply $f(d) = 2df_A(d^2)$:

$$f(d) = \frac{2d}{s^2}\begin{cases} \frac{d^2}{4s^2} & 0 \le d \le s \\ \frac{2d}{s} - \frac{3d^2}{4s^2} - 1 s \le d \le \sqrt{2}s & \\ 2tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}} + \frac{d^2}{4s^2} + \frac{2d}{s} - 4\frac{\sqrt{d^2-s^2}}{s} + 1 & \sqrt{2}s \le d \le 2s \\ 2tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}} + \frac{3d^2}{4s^2} - 4\frac{\sqrt{d^2-s^2}}{s} + 3 & 2s \le d \le \sqrt{5}s \\ 2tan^{-1}\frac{4s^2-\frac{d^2}{2}}{2s\sqrt{d^2-4s^2}} + 2\frac{\sqrt{d^2-s^2}}{s} - \frac{d^2}{4s^2} - 2 & \sqrt{5}s \le d \le \sqrt{8}s \\ 0 & otherwise \end{cases} \tag{11}$$

C. Two Random Points in Parallel Grids (Uniform Size)

In Fig. 4, P and Q are in two vertically or horizontally adjacent grids. Therefore, $x_1, y_1, y_2 \sim U(0,s)$, and $x_2 \sim U(-s,0)$. In this case the pdf of $v_1 = x_1 - x_2$ and $v_2 = y_1 - y_2$ follow different distributions according to equations 9 and III-A respectively:

$$\begin{aligned} f_{V_1}(v) &= \tfrac{v-2(v-s)H(v-s)}{s^2} \quad and \\ f_{V_2}(v) &= \tfrac{v+s-2vH(v)}{s^2} \end{aligned} \tag{12}$$

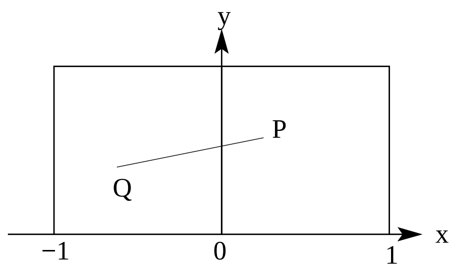


Fig. 4. Random Points in Parallel Grids (Uniform Size).

The square of v_1 and v_2:

$$f_{A_1}(a) = \frac{2H(s-\sqrt{a})-1}{2s^2} + \frac{H(\sqrt{a}-s)}{s\sqrt{a}} \quad and$$
$$f_{A_2}(a) = \frac{1}{s\sqrt{a}} + \frac{1-2H(\sqrt{a})}{s^2}$$

By doing the same convolution process the following is pdf $f(d)$:

$$f(d) = \frac{2d}{s^2} \begin{cases} \frac{d}{s} - \frac{d^2}{2s^2} & 0 \le d \le s \\ tan^{-1}\frac{s\sqrt{d^2-s^2}}{s^2-\frac{d^2}{2}} + \frac{d^2}{s^2} - 2\frac{d+\sqrt{d^2-s^2}}{s} + \frac{3}{2} & s \le d \le \sqrt{2}s \\ tan^{-1}\frac{s\sqrt{d^2-s^2}}{\frac{d^2}{2}-s^2} - 2\frac{d+\sqrt{d^2-s^2}}{s} - \frac{1}{2} & \sqrt{2}s \le d \le 2s \\ tan^{-1}\frac{4s^2-\frac{d^2}{2}}{2s\sqrt{d^2-4s^2}} - tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}} + \frac{\sqrt{d^2-4s^2}+2\sqrt{d^2-s^2}}{s} - \frac{d^2}{2s^2} - \frac{5}{2} & 2s \le d \le \sqrt{5}s \\ 0 & otherwise \end{cases} \tag{13}$$

D. Simulation Results

The above derivation is flexible enough to be scaled and applied to all other grid sizes and non-uniform grid size. Figure 5 shows the comparison between the CDF of distance distribution in two routing schemes (diagonal-first routing and Manhattan walk), and the actual simulation result. In the simulation, both of the routings use unit grid size. As can be observed from the figure, the CDFs and simulation results match very closely.

Figure 6 (a) and (b) show the comparison of two routing schemes, using average distance and distance distribution respectively. The distance distribution matches simulation results accurately, while using average distance, the difference between analysis model and simulation becomes much

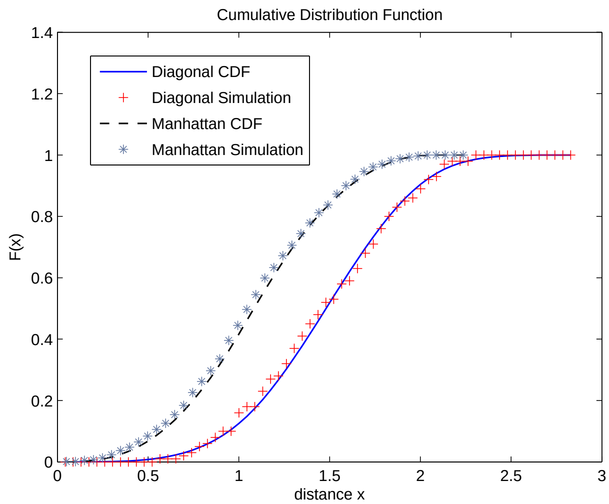
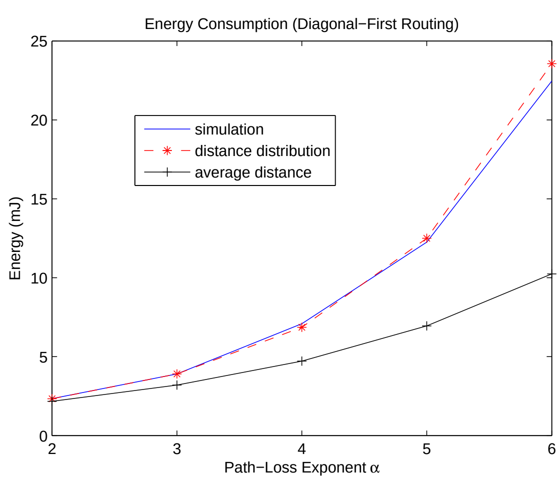
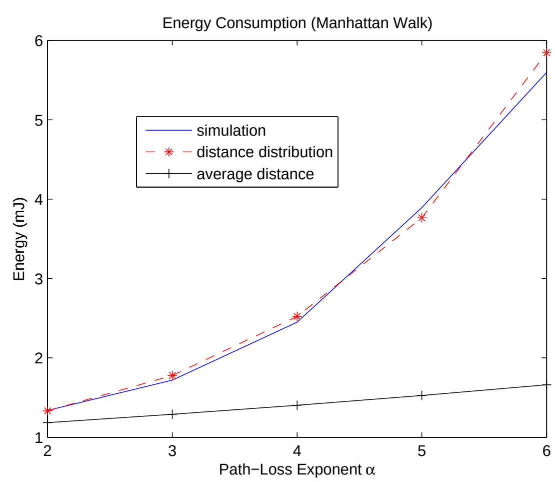


Fig. 5. Cumulative Distribution Function.

larger as the path-loss exponent α becomes bigger. Therefore, although computing pdf is much more difficult, the resulting model characterizes the real network better.



(a)



(b)

Fig. 6. Diagonal-first Routing and Manhattan Walk.

IV. Distance Distribution With Variable Size

As described in section I, using variable grid size will help conserve more energy and prolong network lifetime. In this section we discuss two cases, diagonal and parallel. The distribution results in the case where nodes are located in a single grid in section III-A can still be applied since the length of of the sides can be scaled arbitrarily.

A. Two Random Points in Diagonal Grids (Variable Size)

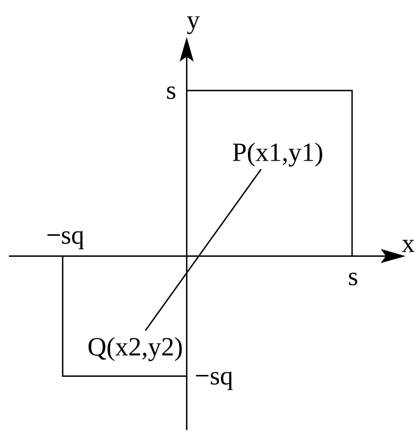


Fig. 7. Two Random Points in Diagonal Grids (Variable Size).

Assume the ratio between the length of the sides of two diagonally adjacent grids is q. In figure 7, $x_1, y_1 \sim U[0, s]$ and $x_2, y_2 \sim U[-sq, 0]$. The difference distribution $v = x_1 - x_2$:

$$f_V(v) = \frac{(v - sq)H(sq - v) + (v - s)H(s - v) + (1 + q)s - v}{s^2 q} \tag{14}$$

The product distribution $a = v^2$:

$$f_A(a) = \frac{(\sqrt{a} - sq)H(sq - \sqrt{a}) + (\sqrt{a} - s)H(s - \sqrt{a}) + (1 + q)s - \sqrt{a}}{2s^2 q\sqrt{a}} \tag{15}$$

Depending on the value of q, the convolution will get different overlapping range. The following is an example when $\sqrt{2}-1 \leq q \leq \frac{1}{\sqrt{2}}$ (other ranges of q include $0 \leq q \leq \frac{\sqrt{3}-1}{2}$, $\frac{\sqrt{3}-1}{2} \leq q \leq \sqrt{2}-1$ and $\frac{1}{\sqrt{2}} \leq q \leq 1$):

$$f(d)=\frac{d}{s^2}\begin{cases}\frac{d^2}{2s^2q^2} & 0\le d\le sq\\ \frac{2d}{sq}-\frac{d^2}{2s^2q^2}-1 & sq\le d\le\sqrt{2}sq\\ tan^{-1}\frac{\frac{d^2}{2}-s^2q^2}{sq\sqrt{d^2-s^2q^2}}+\frac{2}{sq}\left(d-\sqrt{d^2-s^2q^2}\right) & \sqrt{2}sq\le d\le s\\ tan^{-1}\frac{\frac{d^2}{2}-s^2q^2}{sq\sqrt{d^2-s^2q^2}}+\frac{2}{sq^2}\left((1+q)d-q\sqrt{d^2-s^2q^2}\right) & \\ -\frac{d^2}{s^2q^2}-\frac{1}{q^2} & s\le d\le\sqrt{1+q^2}s\\ \frac{1}{q}\left[tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}}+(1+q)tan^{-1}\frac{\frac{d^2}{2}-s^2q^2}{sq\sqrt{d^2-s^2q^2}}\right] & \\ +2\frac{1+q}{sq^2}(d-\sqrt{d^2-s^2q^2})-2\frac{\sqrt{d^2-s^2}}{sq}+1 & \sqrt{1+q^2}\le d\le\sqrt{2}s\\ \frac{1+q}{q^2}\left[tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}}+qtan^{-1}\frac{\frac{d^2}{2}-s^2q^2}{sq\sqrt{d^2-s^2q^2}}\right] & \\ +2\frac{1+q}{sq^2}(d-\sqrt{d^2-s^2q^2}-\sqrt{d^2-s^2})+\frac{d^2}{2s^2q^2}+\frac{1+q^2}{q^2} & \sqrt{2}s\le d\le(1+q)s\\ \frac{1+q}{q^2}\left[tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}}+(qtan^{-1}\frac{\frac{d^2}{2}-s^2q^2}{sq\sqrt{d^2-s^2q^2}}\right] & (1+q)s\le d\le\\ -2\frac{1+q}{sq^2}(\sqrt{d^2-s^2q^2}+\sqrt{d^2-s^2})+\frac{3d^2}{2s^2q^2}+2\frac{1+q+q^2}{q^2} & \sqrt{q^2+(1+q)^2}s\\ \frac{1+q}{q^2}\left[tan^{-1}\frac{\frac{d^2}{2}-s^2}{s\sqrt{d^2-s^2}}+(qtan^{-1}\frac{s^2(1+q)^2-\frac{d^2}{2}}{s(1+q)\sqrt{d^2-s^2(1+q)^2}}\right] & \\ +\frac{2}{sq^2}\left[q\sqrt{d^2-s^2(1+q)^2}-(1+q)\sqrt{d^2-s^2}\right] & \sqrt{q^2+(1+q)^2}s\le d\\ +\frac{d^2}{2s^2q^2}+\frac{1}{q^2} & \le\sqrt{1+(1+q)^2}s\\ \frac{(1+q)^2}{q^2}tan^{-1}\frac{s^2(1+q)^2-\frac{d^2}{2}}{s(1+q)\sqrt{d^2-s^2(1+q)^2}}-\frac{d^2}{2s^2q^2} & \sqrt{1+(1+q)^2}s\le d\\ -\frac{1+q}{sq^2}\left[(1+q)s-2\sqrt{d^2-s^2(1+q)^2}\right] & \le\sqrt{2}(1+q)s\\ 0 & otherwise\end{cases}\tag{16}$$

B. Two Random Points in Parallel Grids (Variable Size)

Because of the grid structure in Fig. 1, in figure 8 the ratio between a and b is $\frac{1-q}{q}$. This is the most complicated case, where $x_1\sim U[0,a]\sim U\left[0,\frac{1-q}{q}b\right]$, $x_2\sim U[-aq,0]\sim U[-(1-q)b,0]$, $y_1\sim U[0,b]$ and $y_2\sim U[0,bq]$. Therefore,

$$\begin{aligned}f_{V_1}(v)&=\frac{q[v-(1-q)b]H[(1-q)b-v]+q(1-q)b+q\left[\frac{1-q}{q}b-v\right]H\left[v-\frac{1-q}{q}b\right]}{(1-q)^2b^2}\quad and\\ f_{V_2}(v)&=\frac{v+bq-vH(v)+[(1-q)b-v]H[v-(1-q)b]}{b^2q}\end{aligned}\tag{17}$$

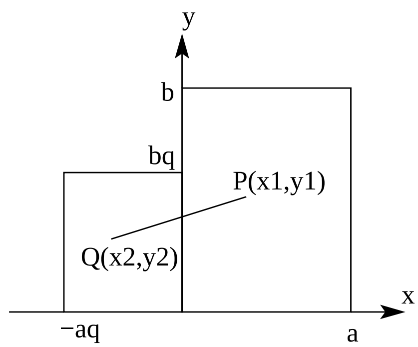


Fig. 8. Two Random Points in Parallel Grids (Variable Size).

The product distribution $s_1 = v_1^2$ is thus:

$$f_{S_1}(s) = \frac{\left[\sqrt{s} - q(1-q)b\right] H\left[(1-q)b - \sqrt{s}\right] + q(1-q)b + \left[(1-q)b - q\sqrt{s}\right] H\left[\sqrt{s} - \left(\frac{1-q}{q}\right) b\right]}{2(1-q)^2 b^2 \sqrt{s}} \tag{18}$$

$$f_{S_2}(s) = \frac{1}{2b^2 q\sqrt{s}} \begin{cases} \left[(1-q)b - \sqrt{s}\right] H\left[\sqrt{s} - (1-q)b\right] & \\ +qb + (qb - \sqrt{s})H(qb - \sqrt{s}) & 0 < q \leq \frac{1}{2} \\ \left[\sqrt{s} - (1-q)b\right] H\left[(1-q)b - \sqrt{s}\right] + b - \sqrt{s} & \\ +(qb - \sqrt{s})H(qb - \sqrt{s})) & \frac{1}{2} \leq q < 1 \end{cases} \tag{19}$$

i.e., when $q \leq \frac{1}{2}$,

$$f_{S_2^1}(s) = \begin{cases} \frac{1}{b\sqrt{s}} - \frac{1}{2b^2 q} & 0 \leq s \leq b^2 q^2 \\ \frac{1}{2b\sqrt{s}} & b^2 q^2 \leq s \leq (1-q)^2 b^2 \\ \frac{1}{2bq\sqrt{s}} - \frac{1}{2b^2 q} & (1-q)^2 b^2 \leq s \leq b^2 \\ 0 & otherwise \end{cases}$$

When $q \geq \frac{1}{2}$,

$$f_{S_2^2}(s) = \begin{cases} \frac{1}{b\sqrt{s}} - \frac{1}{2b^2q} & 0 \le s \le (1-q)^2b^2 \\ \frac{1+q}{2bq\sqrt{s}} - \frac{1}{b^2q} & (1-q)^2b^2 \le s \le b^2q^2 \\ \frac{1}{2bq\sqrt{s}} - \frac{1}{2b^2q} & b^2q^2 \le s \le b^2 \\ 0 & otherwise \end{cases}$$

Sice this is too complicated to derive the pdf, especially for convolution in step 3), we use the approximate distance distribution in the next section.

C. Two Random Points in Parallel Grids (Approximate Distance Distribution)

Since derivation of exact pdf is too complicated, we approximate the actual distance distribution in parallel case as from an outside ring to an inside ring in Fig. 1 (a). Recall in Fig. 1 (b) nodes in parallel rectangles 2 and 4, e.g., R, its coordinate distribution can be formulated as

$$\begin{cases} f_{X_1}(x) = \frac{H(x+q)-H(x)}{q} \\ f_{Y_1}(x) = \frac{H(x)-H[x-(1-q)]}{1-q} \end{cases}, \tag{20}$$

and for S:

$$\begin{cases} f_{X_2}(x) = \frac{H(x+q)-H(x+q(1-q))}{q^2} \\ f_{Y_2}(x) = \frac{H[x+q(1-q)]-H(x)}{q(1-q)} \end{cases}, \tag{21}$$

i.e., $X_1 \sim U[-q, 0]$, $Y_1 \sim U[0, 1-q]$, and $X_2 \sim U[-q, -q(1-q)]$, $Y_2 \sim U[-q(1-q), 0]$. As shown in Fig. 9 (after normalizing to unit size as in Fig. 1). Therefore,

$$\begin{aligned} f_{V_1}(v) &= \frac{-vH(v)+[q(1-q)+v]H[q(1-q)-v]+q}{q^3} \qquad and \\ f_{V_2}(v) &= \frac{[v-q(1-q)]H[q(1-q)-v]+[v-(1-q)]H[(1-q)-v]+(1-q)(1+q)-v}{q(1-q)^2} \end{aligned} \tag{22}$$

Product distribution $s_1 = v_1^2$ and $s_2 = v_2^2$ are:

$$f_{S_1}(s) = \frac{1}{2q^3\sqrt{s}} \begin{cases} \left[\sqrt{s}-q^2\right] H\left[\sqrt{s}-q^2\right] + q - \sqrt{s} & \\ +q(2q-1)H\left[q(1-q)-\sqrt{s}\right] & 0 < q \le \frac{1}{2} \\ \left[q(1-q)-\sqrt{s}\right] H\left[\sqrt{s}-q(1-q)\right] + q - \sqrt{s} & \\ +q(2q-1)H(q^2-\sqrt{s}) & \frac{1}{2} \le q < 1 \end{cases} \tag{23}$$

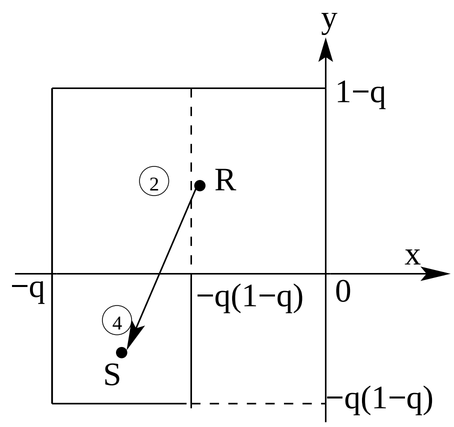


Fig. 9. Two Random Points in Parallel Grids.

i.e., when $q \le \frac{1}{2}$,

$$f_{S_1^1}(s) = \begin{cases} \frac{1}{q\sqrt{s}} - \frac{1}{2q^3} & 0 \le s \le q^4 \\ \frac{1}{2q\sqrt{s}} & q^4 \le s \le q^2(1-q)^2 \\ \frac{1}{2q^2\sqrt{s}} - \frac{1}{2q^3} & q^2(1-q)^2 \le s \le q^2 \\ 0 & otherwise \end{cases}$$

When $q \ge \frac{1}{2}$,

$$f_{S_1^2}(s) = \begin{cases} \frac{1}{q\sqrt{s}} - \frac{1}{2q^3} & 0 \le s \le q^2(1-q)^2 \\ \frac{1+q}{2q^2\sqrt{s}} - \frac{1}{q^3} & q^2(1-q)^2 \le s \le q^4 \\ \frac{1}{2q^2\sqrt{s}} - \frac{1}{2q^3} & q^4 \le s \le q^2 \\ 0 & otherwise \end{cases}$$

$$f_{S_2}(s) = \frac{\left[\sqrt{s} - q(1-q)\right] H\left[q(1-q) - \sqrt{s}\right] + \left[\sqrt{s} - (1-q)\right] H\left[(1-q) - \sqrt{s}\right] + (1-q)(1+q) - \sqrt{s}}{2q(1-q)^2\sqrt{s}} \tag{24}$$

i.e.,

$$f_{S_2}(s)=\begin{cases}\frac{1}{2q(1-q)^2} & 0\le s\le q^2(1-q)^2\\ \frac{1}{2(1-q)\sqrt{s}} & q^2(1-q)^2\le s\le (1-q)^2\\ \frac{1+q}{2q(1-q)\sqrt{s}}-\frac{1}{2q(1-q)^2} & (1-q)^2\le s\le (1-q)^2(1+q)^2\\ 0 & otherwise\end{cases}$$

As the diagonal case, parallel distribution also depends on the value of q. The following is an example when $(1-\frac{1}{\sqrt{2}})\le q\le\frac{\sqrt{2}}{3}$ (other ranges of q include $0\le q\le 1-\frac{1}{\sqrt{2}}$, $\frac{\sqrt{2}}{3}\le q\le\frac{1}{2}$, $\frac{1}{2}\le q\le 2-\sqrt{2}$, $2-\sqrt{2}\le q\le\frac{\sqrt{5}-1}{2}$, $\frac{\sqrt{5}-1}{2}\le q\le\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}\le q\le 1$):

$$f(d)=\begin{cases}\frac{2d^2}{q^2(1-q^2)}-\frac{d^3}{2q^4(1-q)^2} & 0\le d\le q^2\\ \frac{d^2}{q^2(1-q^2)}+\frac{d}{2(1-q)^2} & 0\le d\le q(1-q)\\ \frac{d}{q(1-q)}sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)+\frac{d}{q^2(1-q)^2}\left[d-2\sqrt{d^2-q^2(1-q)^2}\right] & \\ \quad+\frac{(1-q)\pi d+qd}{2q(1-q)^2} & q(1-q)\le d\le q\sqrt{(1-q)^2+q^2}\\ \frac{d}{2q(1-q)}\left[sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)-sin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right]-\frac{d^3}{2q^4(1-q)^2} & \\ \quad+\frac{d}{q^3(1-q)^2}\left[qd-q\sqrt{d^2-q^2(1-q)^2}+(1-q)\sqrt{d^2-q^4}\right] & \\ \quad+\frac{q\pi d-(1-q)d}{2q^2(1-q)} & q\sqrt{(1-q)^2+q^2}\le d\le\sqrt{2}q(1-q)\\ \frac{d}{2q^2(1-q)}\left[(2-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)-qsin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\ \quad+\frac{d}{q^3(1-q)^2}\left[qd-(2-q)\sqrt{d^2-q^2(1-q)^2}+(1-q)\sqrt{d^2-q^4}\right] & \\ \quad+\frac{\pi d}{2q(1-q)}+\frac{d}{2q^2} & \sqrt{2}q(1-q)\le d\le q\\ \frac{d}{2q^2(1-q)}\left[(2-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)-qsin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\ \quad+\frac{d}{q^3(1-q)^2}\left[(1-q)\sqrt{d^2-q^4}-(2-q)\sqrt{d^2-q^2(1-q)^2}\right] & \\ \quad+\frac{d^3}{2q^4(1-q)^2}+\frac{\pi d}{2q(1-q)}+\frac{(q^2-2q+2)d}{2q^2(1-q)^2}-\frac{d^2}{q^3(1-q)} & q\le d\le q\sqrt{(1-q)^2+1}\end{cases} \tag{25}$$

(continue)

$$
f(d)=\begin{cases}
\frac{d}{2q^2(1-q)}\left[(1-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)-qsin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\
\quad+\frac{d}{q^3(1-q)^2}\left[\sqrt{d^2-q^4}+\sqrt{d^2-q^2}-d-\sqrt{d^2-q^2(1-q)^2}\right] & \\
\quad-\frac{d}{2q^2(1-q)}sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)+\frac{\pi d}{2q(1-q)} & q\sqrt{(1-q)^2+1}\leq d\leq(1-q) \\
\frac{d}{2q^2(1-q)}\left[(1-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)+2sin^{-1}\left(\frac{d^2-2(1-q)^2}{d^2}\right)\right] & \\
\quad-\frac{d}{2q^2(1-q)}\left[sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)+qsin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\
\quad+\frac{d}{q^3(1-q)}\left[\sqrt{d^2-q^2}+\sqrt{d^2-q^4}-\sqrt{d^2-q^2(1-q)^2}\right] & \\
\quad-\frac{d}{q^4(1-q)^2}\left[(1-q^2)d+2q^2\sqrt{d^2-(1-q)^2}\right] & \\
\quad+\frac{d^3}{2q^4(1-q)^2}+\frac{(1+q)\pi d}{2q^2(1-q)}+\frac{d}{2q^4} & (1-q)\leq d\leq\sqrt{(1-q)^2+q^4} \\
\frac{d}{2q^2(1-q)}\left[(1-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)+sin^{-1}\left(\frac{d^2-2(1-q)^2}{d^2}\right)\right] & \\
\quad-\frac{d}{2q^2(1-q)}\left[sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)+(1+q)sin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\
\quad-\frac{d}{q^3(1-q)^2}\left[(1-q)\sqrt{d^2-q^2(1-q)^2}+q\sqrt{d^2-(1-q)^2}\right] & \\
\quad+\frac{(1+q)d}{q^4(1-q)}\left(\sqrt{d^2-q^2}+\sqrt{d^2-q^4}-d\right)+\frac{(1+q)\pi d}{2q^2(1-q)}-\frac{d}{2(1-q)^2} & \sqrt{(1-q)^2+q^4}\leq d\leq(1-q)\sqrt{q^2+1} \\
\frac{d}{2q^3(1-q)}\left[(1-q^2)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)+sin^{-1}\left(\frac{d^2-2(1-q)^2}{d^2}\right)\right] & \\
\quad-\frac{d}{2q^2(1-q)}\left[sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)+(1+q)sin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\
\quad+\frac{d}{q^3(1-q)^2}\left[(1-q)\sqrt{d^2-q^2}-\sqrt{d^2-(1-q)^2}\right] & \\
\quad+\frac{(1+q)d}{q^4(1-q)}\left[\sqrt{d^2-q^4}-\sqrt{d^2-q^2(1-q)^2}-d\right] & \\
\quad+\frac{d^3}{2q^4(1-q)^2}+\frac{(1+q)\pi d}{2q^2(1-q)}+\frac{(1-2q+2q^2-2q^3)d}{2q^4(1-q)^2} & (1-q)\sqrt{q^2+1}\leq d\leq\sqrt{(1-q)^2+q^2} \\
\frac{(1+q)d}{2q^3(1-q)}\left[(1-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right)-qsin^{-1}\left(\frac{d^2-2q^4}{d^2}\right)\right] & \\
\quad+\frac{(1+q)d}{q^4(1-q)}\left[\sqrt{d^2-q^2}+\sqrt{d^2-q^4}-\sqrt{d^2-q^2(1-q)^2}-d\right] & \\
\quad-\frac{(1+q)d}{2q^3(1-q)}sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)+\frac{(1-q^2)\pi d-2qd}{2q^2(1-q)^2} & \sqrt{(1-q)^2+q^2}\leq d\leq(1-q^2)
\end{cases}
\tag{26}
$$

(continue)

$$f(d) = \begin{cases} \frac{(1+q)d}{2q^3(1-q)}\left[(1-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right) - sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)\right] & \\ \quad - \frac{(1+q)d}{2q^2(1-q)}\left[sin^{-1}\left(\frac{d^2-2q^4}{d^2}\right) + sin^{-1}\left(\frac{d^2-2(1-q^2)^2}{d^2}\right)\right] & \\ \quad + \frac{(1+q)d}{q^4(1-q)}\left[\sqrt{d^2-q^2} - \sqrt{d^2-q^2(1-q)^2} + \sqrt{d^2-q^4}\right] & \\ \quad - \frac{d^3}{2q^4(1-q)^2} + \frac{d\sqrt{d^2-(1-q)^2(1+q)^2}}{q^2(1-q)^2} - \frac{(q^4+2q^3-2q^2+1)d}{2q^4(1-q)^2} & (1-q^2) \le d \le \sqrt{(1-q^2)^2+q^4} \\ \frac{(1+q)d}{2q^3(1-q)}\left[(1-q)sin^{-1}\left(\frac{d^2-2q^2(1-q)^2}{d^2}\right) - sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)\right] & \\ \quad - \frac{(1+q)d}{2q^2(1-q)}sin^{-1}\left(\frac{d^2-2(1-q)^2(1+q)^2}{d^2}\right) - \frac{(2-q)d}{2q(1-q)^2} & \\ \quad + \frac{(1+q)d}{q^4(1-q)}\left[\sqrt{d^2-q^2} - \sqrt{d^2-q^2(1-q)^2}\right] & \\ \quad + \frac{d\sqrt{d^2-(1-q)^2(1+q)^2}}{q^2(1-q)^2} & \sqrt{(1-q^2)^2+q^4} \le d \le (1-q)\sqrt{(1+q)^2+q^2} \\ \frac{(1+q)d}{2q^3(1-q)}\left[sin^{-1}\left(\frac{2(1-q)^2(1+q)^2-d^2}{d^2}\right) - sin^{-1}\left(\frac{d^2-2q^2}{d^2}\right)\right] & \\ \quad + \frac{d\sqrt{d^2-(1-q)^2(1+q)^2}}{q^3(1-q)^2} + \frac{d(1+q)\sqrt{d^2-q^2}}{q^4(1-q)} - \frac{(q^4-q^2+1)d+d^3}{2q^4(1-q)^2} & (1-q)\sqrt{(1+q)^2+q^2} \le d \le \sqrt{(1-q^2)^2+q^2} \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

V. Wireless Channel & Traffic Model

A. Energy Consumption Between Transceivers

According to [6], the energy consumed by radio transmitter is

$$E_{\mathrm{Tx}} = \lambda\epsilon \int x^{\alpha} f_D(x)dx, \tag{28}$$

where λ is the data transmission rate, ϵ is a constant related to the environment, and α is the path loss exponent with values from 2 to 6. The exact form of $f_D(x)$ is the pdf of either diagonal or parallel distance distribution.

B. Accumulated Many-To-One Traffic

In wireless sensor networks, each sensor generates data at a rate λ bits/second. The data is transmitted from the source node to its cluster head, which then forwards the data to the sink through other cluster heads. This multi-hop forwarding leads to the many-to-one traffic pattern: in

Fig. 1 (a), the cluster heads in the i-th ring receive both the traffic originating from its own cluster, as well as the traffic relayed from the $(i-1)$-th ring, then they forward the combined traffic to the cluster heads in the $(i+1)$-th ring.

C. Energy Optimization

Let E_i be the total communication energy used by cluster heads in the i-th ring, then

$$E_i = \lambda_i (E_{\mathrm{Re}} + E_{\mathrm{Te}}) + \lambda_i E_{\mathrm{Tx}} \tag{29}$$

where λ_i is the data rate passing through the i-th ring, including the data rate from the sensors to their corresponding cluster heads, and the data rate between cluster heads in neighboring rings. Here we assume the data generated by cluster heads in the i-th ring is part of the traffic that realyed through the i-th ring. E_{Re} and E_{Te} are the energy consumed by transceiver circuitry. So the first half of (29) is the energy consumed by electrical circuit, while the second half is for radio communication. According to [5], we have $E_{\mathrm{Re}} = E_{\mathrm{Te}} = E_e$, then E_i is given by

$$E_i = \lambda_i (2E_e + E_{\mathrm{Tx}}) \tag{30}$$

λ_i is essentially the bit rate of aggregate traffic that originates from the grids in the most outside ring, through the i-th ring. Since the sensing field is symetric, the aggregated data can be divided into two categories as in Fig. 1 (a): between parallel rectangles in both the upper and lower half of the field (λ_{ip}), and between diagonal squares in a "straight line" (λ_{id}), i.e., $\lambda_i = 2\lambda_{ip} + \lambda_{id}$. Suppose the node density is ρ, then

$$\begin{cases} \lambda_{ip} = \frac{A^2 q}{1+q}(1-q^{2i})\rho\lambda \\ \lambda_{id} = \frac{A^2(1-q)}{1+q}(1-q^{2i})\rho\lambda \end{cases} . \tag{31}$$

Therefore,

$$\begin{aligned} E_i = {} & 2\lambda_{ip}\left(2E_e + \epsilon \int x^{\alpha} f_{D_{\mathrm{P}}}(x)dx\right) \\ & + \lambda_{id}\left(2E_e + \epsilon \int x^{\alpha} f_{D_{\mathrm{D}}}(x)dx\right). \end{aligned} \tag{32}$$

To minimize the total network energy consumption, we formulate the problem as

$$\begin{aligned} \min \quad & \textstyle\sum_{i=1}^{K} E_i \\ s.t. \quad & Aq^{K-1} > r_0 \\ & Aq^{K} <= r_0 \end{aligned} \tag{33}$$

where r_0 is the minimal distance between wireless transceivers, thus K is the maximum number of hops from the source node to the sink.

Meanwhile, the maximization of network lifetime is equivalent to the following formulation:

$$\min \quad \max_i E_i \tag{34}$$

where $i = 1...K^2$, so E_i is calculated for each grid.

VI. Simulation With Variable Grid Size

1) Channel Propagation Model: In the simulation experiment, both free space model and multi-path fading model were used, depending on the distance between the transmitter and receiver. According to [5], if this distance is less than a certain cross-over distance d_c, then the Friis free space model is used; otherwise the two-ray ground propagation will be applied instead. d_c is defined as:

$$d_c = \frac{4\pi\sqrt{L}h_r h_t}{\lambda} \tag{35}$$

where $L \geq 1$ is the system loss factor, h_r and h_t are the height of receiver/transmitter antenna, and λ is the wavelength of the carrier signal.

Then energy consumption between two transceivers with distance d will be:

$$E_{Tx} = \begin{cases} \epsilon_{Friis}\lambda d^2 & d \leq d_c \\ \epsilon_{two-ray}\lambda d^4 & d \geq d_c \end{cases}$$

where λ is the radio bitrate.

[5] defines how to calculate the values of coefficients ϵ_{Friis} and $\epsilon_{two-ray}$. In our simulation, we used the following parameters: $h_t = h_r = 0.6\ m$, no system loss ($L = 1$), $2.4\ GHz$ radio

frequency[1], and $\lambda = \frac{3\times10^8}{2.4\times10^9} = 0.125\ m$. Plugging these into equation 35, $d_c = 36.2\ m$.

2) PDF and Polynormial Fitting: Figure 10 shows the probability distribution functions and their polynormial fitting in the case where both Diagonal-First and Manhattan Walk has a grid size ratio of 0.5. From the figure it is clear that normal distribution can not be a good approximation.

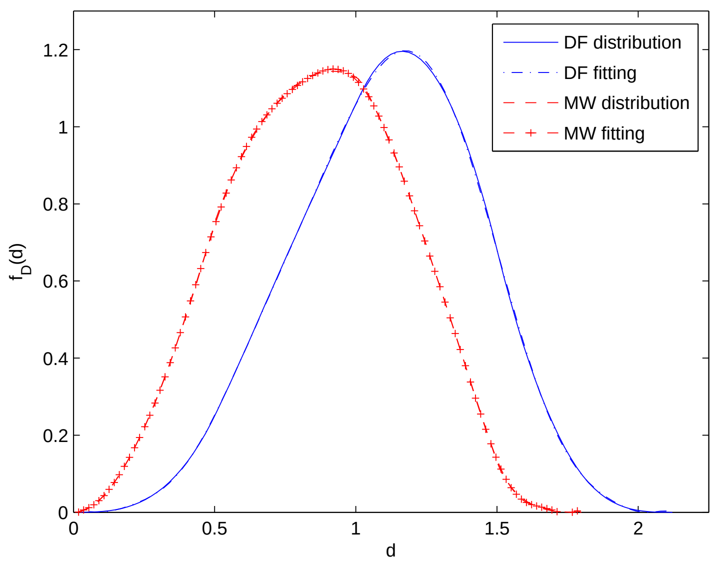


Fig. 10. Simulation With Variable Grid Size.

A. Distance Verification

Due to space limit, we only present the verification of distance distribution for diagonal grids. We generated $1,000$ pairs of random points. Figure 11 shows the cumulative distribution function and simulation result when $q = 0.4$ and $q = 0.7$. The simulation and distribution function match each other quite well, with only slight deviation. This validates the accuracy of our derivation and energy consumption model. The dashed lines in Fig. 11 are distribution average (where the value of CDF is 0.5) while the dotted lines are the min-max average. The difference between them is innegligible, especially when path loss coefficient gets higher.

[1]According to IEEE $802.15.4$ specification, ZigBee operates in the industrial, scientific and medical (ISM) radio bands; $868\ MHz$ in Europe, $915\ MHz$ in the USA and Australia, and $2.4\ GHz$ in most jurisdictions worldwide.

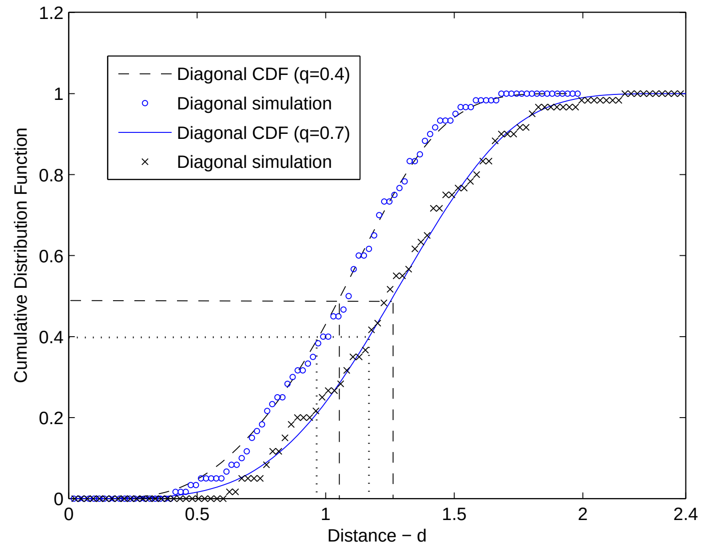


Fig. 11. Numerical and Simulation Results For Distance Distribution.

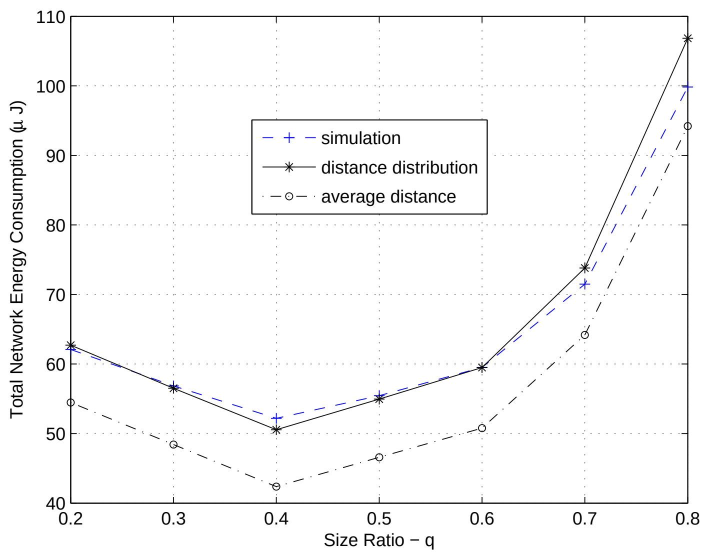


Fig. 12. Total Network Energy Consumption.

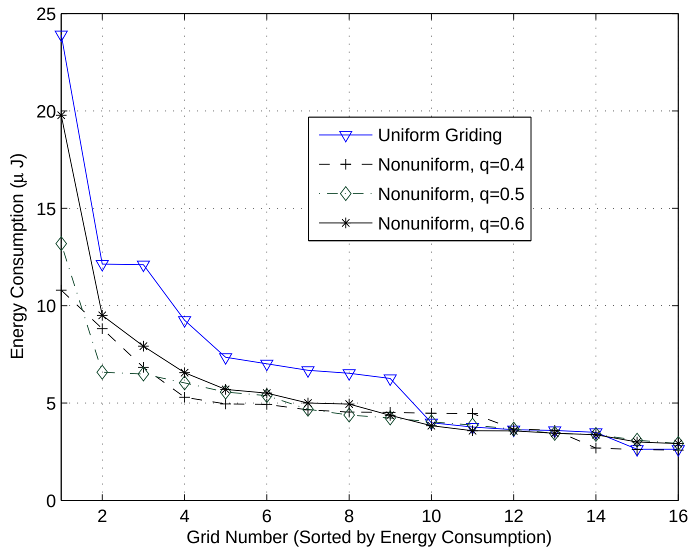


Fig. 13. Per-Grid Energy Consumption with Nonuniform Griding.

As in Fig. 12, the energy consumption model using distance distribution matches the simulation results with high accuracy, while the model using average node distance largely underestimates the real value. There also exists an optimal size ratio q between 0.3 and 0.5, that minimizes the total network energy consumption.

Figure 13 shows the effect of energy balancing of nonuniform griding. The energy consumption per grid is sorted in the descending order for each grid. Compared with the results from uniform griding, it is obvious that nonuniform griding with a proper grid size ratio can reduce the maximum energy consumption, thus balancing the overall energy distribution. According to the value of q in the figure, network energy is more balanced when q is between 0.3 and 0.5.

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