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A Geometric Probability Model for Capacity Analysis and Interference Estimation in Wireless Mobile Cellular Systems

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***Abstract*—Performance metrics in cellular systems, such as per-user link capacity and co-channel interference, are dependent on the statistical distances between communicating nodes. An analytical model based on geometric probability in cellular systems is presented here for capacity analysis and interference estimation. We first derive the closed-form distance distribution between cellular base stations and mobile users, giving the explicit probability density functions of the distance from a base station to an arbitrary user in the same hexagonal cell, or to the users in adjacent cells. Different from numerical methods or approximation, and the existing approaches in geometric probability, this unified approach provides explicit distribution functions that can lead to all statistical moments, and is not limited by coordinate distributions, either of base stations or subscribers. Analytical results on per-user link capacity and co-channel interference are derived and validated through simulation, which shows the high accuracy and promising potentials of this approach.**

***Index Terms*—Geometric probability, distance distribution, cellular systems, hexagon, capacity, interference**

I. Introduction

Spatial reuse is a fundamental enabling tool to achieve a higher network capacity for the ever-increasing demand for wireless services. A wireless communication network under concern is usually divided into congruent polygons, or cells. Examples exist for equilateral triangles, squares, and regular hexagons. While squares are widely used in ad hoc networks [1], hexagons are typical in cellular systems as they provide the most economic coverage of the network, without leaving gaps between cells. Our focus in this paper is hexagonal cells, where users communicate via a fixed base station (BS) located in a cell.

In a cellular system, both the signal and interference strength decay super-linearly with distance. Therefore, the distance distribution between the BS and mobile users is a key element to quantify the fundamental network performance metrics and to estimate the system capacity, which is critical for cell planning and resource management. Also, the random distribution of mobile users also models the spatial distribution of the traffic generated by the users in the cell. The distance between users is thus highly important, which calls for an accurate model that can lead to all statistical characteristics of a cellular system. However, although the geometric probability model based on distance distributions has been developed and applied to the area of city planning and transportation [2], forestry and chemistry [3], relatively little research has appeared in the field of wireless communications.

Many distance models simply used a min-max average distance, i.e., by averaging minimum and maximum values. However, min-max average is meaningful only when the actual distance distribution is symmetric. Numerical average distances and distributions through Monte Carlo simulations also exist, and [4]–[8] used the distance distribution for a circle to approximate the distribution for a hexagon. Although using an average value or approximation is simple, the nonlinear path loss with respect to the distance makes the performance metric more complicated. The error of the numerical average and empirical distribution grows exponentially with a super-linear path loss exponent in wireless communications.

In a hexagonal cellular system, the distance distributions between a random user and its BS and the neighboring BSs are a challenging issue, which motivates this work. The main contributions of this paper are twofold. First, the closed-form distance distributions between cellular BSs and mobile users in a hexagonal cellular system are derived, using an *area-ratio* approach that is not limited by the coordinates of mobile users and BSs. The derivation of the distance distributions is simpler when compared with the existing methods in geometric probability, by using a unified approach leading to distance distributions from a BS to users both in the same and neighboring hexagonal cells. Furthermore, by minor extensions, this approach can be applied to deriving the distance distributions from a non-center point, either inside or outside a hexagon, in a practical scenario where BS cannot be deployed in the center of a cell due to construction limitation. Second, we use the closed-form distance distributions to build the analytical models for location-dependent performance metrics, such as per-user link capacity and co-channel interference. We show that our distance distribution models are accurate in analyzing the distributions of important performance metrics, which lead to all statistical moments, while traditional approximations are inaccurate in modeling both the distribution and the average of the performance. We further find that using a circle with the same area as a hexagon provides the best approximation of the hexagon distribution model. The accuracy of the corresponding analysis is verified through extensive simulation, which shows the promising potentials of our approach.

The rest of the paper is organized as follows. In Section II, we briefly review the research on the distance models in wireless networks. In Section III, we present the derivation of probabilistic distance distributions through the area-ratio approach, for random distances from a BS to an arbitrary user inside the same cell, and in adjacent cells. The extension of this model to a non-center (BS) location is also given in this section. The comparison between the simulation and analytical results is given in Section IV, followed by Section V, which concludes the paper and points out the future research issues.

II. Background and Related Work

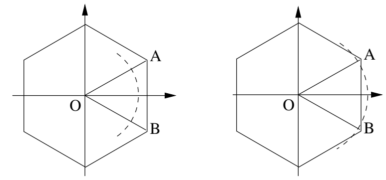
A. Geometry, Distance Distribution and Wireless Networks

Not until recently have people in the field of networking began using geometric and distance distribution models in the analytical modeling and optimization of networking systems. [9] derived the distribution of link distances between two random nodes in a rectangular area. [10] then studied the joint distribution of link distances, i.e., two-hop connectivity, in a square area. [1] investigated the energy consumption in wireless sensor networks by deriving the distance distributions inside and between squares. These models all have provided key insights into our understanding of the statistical characteristics of wireless communication networks, and have given accurate results for wireless protocol design and system dimensioning. While all the above work is done for squares and rectangles, the distance distributions for hexagons are much more useful in a cellular system. However, as the geometric shape of a hexagon is more complicated than a square or rectangle, one can also envision that it is more challenging to obtain the corresponding distance distributions.

B. Hexagons and Cellular Systems

Distance distributions in hexagons are not well-studied in the literature. Strong assumptions on the location distribution of the users, or approximations on the hexagonal geometry were made in order to conveniently analyze performance metrics. In [5], other-cell interference in a multi-cell CDMA system was evaluated, where the hexagonal cells were replace by circles of equal area, and a *fictitious* location PDF where all users in a cell have an equal distance to the base station was assumed. [7] claimed that there is no closed-form interference expression within the same cell, even though it used circular cells to represent hexagons. [6] also used circular cells for approximation, but it was able to give a general formulation of the power received at a base station. As later shown in Section IV, the distance distributions for circles [11] cannot represent that for hexagons, which have very different distance characteristics.

In [3], the distribution function of the distance for a triangle from a vertex was given. Using the symmetric property of a hexagon, [3] can lead to the distribution from a BS to the users in the same cell. However, we show in Section III that a new approach based on the area-ratio is much more mathematically tractable, which can derive the same result as [3]. Moreover, by minor extensions, this approach can be applied to deriving the distance distributions from a non-center point, either inside or outside a cell, to any point inside the hexagonal cell. For instance, the distance from any of the second-layer hexagon centers, and from a hexagon vertex (Section III-C) to the interior of a hexagon, etc.



(a) $0 \le d \le \frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{2} \le d \le 1$

Fig. 1. Random Distances from the Center of a Hexagon.

[12] studied the angle of arrival (AoA) of interfering signals in the uplink of a cellular system, which is given for adjacent hexagonal cells only, and the model is only applicable to interference analysis. [12] also gives the closed-form results within a single hexagon, with the first-order linear approximation. [13] gave an adjacent cell channel interference model which is between two base stations. However, the result in [13] was not given in closed-form. [14] has given the distance distribution from a hexagon center to surrounding hexagonal cells, of which the method was based on [15]: PDF is equal to the arc length inside a hexagon, normalized by the sectional area only if nodes are uniformly distributed. Although this method derives PDF directly, it only applies to uniform distribution.

III. Random Distances Associated with Hexagons

In this section, we give a geometric probability approach based on an area-ratio, by assuming a uniform distribution of users. This approach gives the explicit probability distributions of the distances associated with hexagons, which can be used to further build analytical models in a cellular system.

A. Random Distances from the Center of a Hexagon

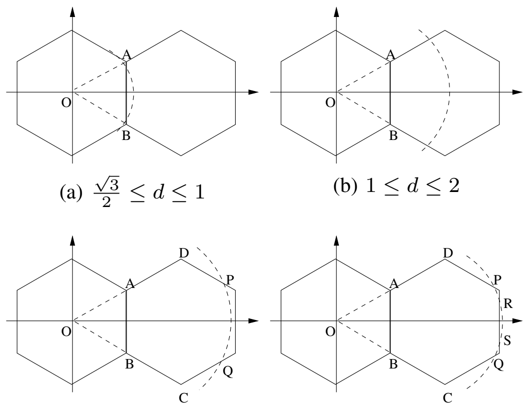
A regular hexagon can be divided into 6 equilateral triangles. The distance distribution from the center of a hexagon to an arbitrary node (in the same hexagon), is equivalent to that from the center to anywhere inside one of the equilateral triangles, e.g., OAB in Fig. 1. Suppose each side of the hexagon has the length of 1 unit, and denote D as the random distance. The probability $P(D \le d)$ is equal to the *ratio* between the area $\mathcal{S}$, where the circle $x^2 + y^2 = d^2$ intersects with triangle OAB, and the area of OAB, $\mathcal{A} = \frac{\sqrt{3}}{4}$. Depending on the value of d, $\mathcal{S}$ has the following two cases.

1) $0 \le d \le \frac{\sqrt{3}}{2}$*:* As shown in Fig. 1(a), the circle $x^2+y^2 = d^2$ is entirely inside the hexagon. Since area $\mathcal{S} = \frac{\pi d^2}{6}$ of the circle intersects with OAB, we have

$$F_D(d) = P(D \le d) = \frac{\mathcal{S}}{\mathcal{A}} = \frac{2\pi}{3\sqrt{3}} d^2. \quad (1)$$

2) $\frac{\sqrt{3}}{2} \le d \le 1$*:* In this case the circle is cut off by the edges of the hexagon, e.g., AB in Fig. 1 (b). The intersection is the area of the sector, $\frac{\pi d^2}{6}$, minus the area that is cut off by AB, i.e., $\mathcal{S} = \frac{\pi d^2}{6} - \left(d^2 \cos^{-1}\frac{\sqrt{3}}{2d} - \frac{\sqrt{3}}{2}\sqrt{d^2 - \frac{3}{4}}\right)$. Therefore,

$$F_D(d) = \frac{2}{\sqrt{3}} \left(\frac{\pi d^2}{3} - 2d^2 \cos^{-1} \frac{\sqrt{3}}{2d} + \sqrt{3}\sqrt{d^2 - \frac{3}{4}} \right). \quad (2)$$



(a) $\frac{\sqrt{3}}{2} \le d \le 1$ (b) $1 \le d \le 2$

(c) $2 \le d \le \frac{3\sqrt{3}}{2}$ (d) $\frac{3\sqrt{3}}{2} \le d \le \sqrt{7}$

Fig. 2. Random Distances from the Center of an Adjacent Hexagon.

Although the distribution function of d in (1)–(2) can also be derived by the approach in [3], the approach used here is much more convenient than the geometric integral in [3]. By $f_D(d) = F'_D(d)$, the probability density function of the random distances inside a hexagon is

$$f_{D_{\mathrm{I}}}(d) = \frac{4d}{\sqrt{3}} \begin{cases} \frac{\pi}{3} & 0 \le d \le \frac{\sqrt{3}}{2} \\ \frac{\pi}{3} - 2\cos^{-1}\frac{\sqrt{3}}{2d} & \frac{\sqrt{3}}{2} \le d \le 1 \\ 0 & \text{otherwise} \end{cases} . \quad (3)$$

B. Random Distances from the Center of an Adjacent Hexagon

The derivation of the distance distribution from the center of an adjacent hexagon also can be treated as an area ratio. Here we need to divide $\mathcal{S}$ by the area of a hexagon, i.e., $\mathcal{A} = \frac{3\sqrt{3}}{2}$. Suppose two hexagons are adjacent to each other as shown in Fig. 2, and let the center of one hexagon be the origin. If each side of the hexagons has an unit length, $\mathcal{S}$ has the following four cases.

1) $\frac{\sqrt{3}}{2} \le d \le 1$*:* $\mathcal{S}$ in this case is the area of the sector cut off by edge AB, as shown in Fig. 2 (a). Then $\mathcal{S} = d^2 \cos^{-1}\frac{\sqrt{3}}{2d} - \frac{\sqrt{3}}{2}\sqrt{d^2 - \frac{3}{4}}$, and

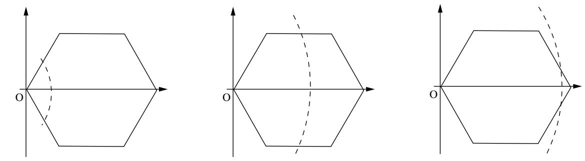
$$F_D(d) = \frac{\mathcal{S}}{\mathcal{A}} = \frac{2}{3\sqrt{3}}\left(d^2\cos^{-1}\frac{\sqrt{3}}{2d} - \frac{\sqrt{3}}{2}\sqrt{d^2 - \frac{3}{4}}\right). \quad (4)$$

2) $1 \le d \le 2$*:* $\mathcal{S}$ is the area of the sector minus the area of triangle OAB, as in Fig. 2 (b). Thus, $\mathcal{S} = \frac{\pi d^2}{6} - \frac{\sqrt{3}}{4}$, and

$$F_D(d) = \frac{2}{3\sqrt{3}}\left(\frac{\pi d^2}{6} - \frac{\sqrt{3}}{4}\right). \quad (5)$$

3) $2 \le d \le \frac{3\sqrt{3}}{2}$*:* As shown in Fig. 2 (c), $\mathcal{S}$ is the sum of trapezoidal area $S_{\mathrm{ABCD}} = \frac{3\sqrt{3}}{4}$, $S_{\mathrm{PQCD}} = \frac{\sqrt{3}}{4}(\sqrt{d^2-3} - 1)(5 - \sqrt{d^2-3})$, and the part that belongs to the sector $S_{\widehat{\mathrm{PQ}}} = d^2\sin^{-1}\frac{3-\sqrt{d^2-3}}{2d} - \frac{\sqrt{3}}{4}(1+\sqrt{d^2-3})(3-\sqrt{d^2-3})$. Then,

$$F_D(d) = \frac{2}{3\sqrt{3}}\left(d^2\sin^{-1}\frac{3-\sqrt{d^2-3}}{2d} + \sqrt{3}\sqrt{d^2-3} - \frac{5\sqrt{3}}{4}\right). \quad (6)$$



(a) $0 \le d \le 1$ (b) $1 \le d \le \sqrt{3}$ (c) $\sqrt{3} \le d \le 2$

Fig. 3. Random Distances from One Vertex of a Hexagon.

4) $\frac{3\sqrt{3}}{2} \le d \le \sqrt{7}$*:* $\mathcal{S}$, as shown in Fig. 2 (d), is the sum of trapezoidal area $S_{\mathrm{ABCD}} = \frac{3\sqrt{3}}{4}$, $S_{\mathrm{PQCD}} = \frac{\sqrt{3}}{4}(\sqrt{d^2-3} - 1)(5 - \sqrt{d^2-3})$, which are the same as the last case, and the part that belongs to the sector but cut off by the edge of hexagon, $S_{\widehat{\mathrm{PR}}\widehat{\mathrm{SQ}}} = d^2\sin^{-1}\frac{3-\sqrt{d^2-3}}{2d} - \frac{\sqrt{3}}{4}(1+\sqrt{d^2-3})(3-\sqrt{d^2-3}) - \left(d^2\cos^{-1}\frac{3\sqrt{3}}{2d} - \frac{3\sqrt{3}}{2}\sqrt{d^2 - \frac{27}{4}}\right)$. Therefore,

$$\begin{aligned} F_D(d) &= \frac{2}{3\sqrt{3}}\left[d^2\left(\sin^{-1}\frac{3-\sqrt{d^2-3}}{2d} - \cos^{-1}\frac{3\sqrt{3}}{2d}\right)\right. \\ &\quad \left. + \sqrt{3}\left(\sqrt{d^2-3} + \frac{3}{2}\sqrt{d^2 - \frac{27}{4}}\right) - \frac{5\sqrt{3}}{4}\right]. \end{aligned} \quad (7)$$

Combining (4)–(7), and by $f_D(d) = F'_D(d)$, the probability density function of random distances, from the center of one hexagon to an arbitrary node in an adjacent hexagon, is

$$f_{D_{\mathrm{A}}}(d) = \frac{4d}{3\sqrt{3}} \begin{cases} \cos^{-1}\frac{\sqrt{3}}{2d} & \frac{\sqrt{3}}{2} \le d \le 1 \\ \frac{\pi}{6} & 1 \le d \le 2 \\ \sin^{-1}\frac{3-\sqrt{d^2-3}}{2d} & 2 \le d \le \frac{3\sqrt{3}}{2} \\ \sin^{-1}\frac{3-\sqrt{d^2-3}}{2d} - \cos^{-1}\frac{3\sqrt{3}}{2d} & \frac{3\sqrt{3}}{2} \le d \le \sqrt{7} \\ 0 & \text{otherwise} \end{cases} . \quad (8)$$

C. Model Extensions

Similarly, the distance distribution from a non-center point, e.g., a vertex of a hexagon, can be derived using the same area-ratio approach. According to Fig. 3(a)–(c), we have the following three subcases:

1) $0 \le d \le 1$*:* As shown in Fig. 3(a), $\mathcal{S} = \frac{\pi d^2}{3}$, and $\mathcal{A} = \frac{3\sqrt{3}}{2}$, therefore,

$$F_D(d) = \frac{2\pi}{9\sqrt{3}}d^2. \quad (9)$$

2) $1 \le d \le \sqrt{3}$*:* The intersection area in this case, as shown in Fig. 3(b), is the sector with radius d and angle $\frac{2\pi}{3}$, minus two parts that extend outside the hexagon on each side. That is, $\mathcal{S} = \frac{\pi d^2}{3} - 2\left(\frac{\theta}{2}d^2 - \frac{\sqrt{3}}{4}\frac{\sqrt{4d^2-3}-1}{2}\right)$, where and $\theta = \sin^{-1}\frac{\sqrt{3}}{4}\frac{\sqrt{4d^2-3}-1}{d}$. By simplification, $\frac{\pi}{3} - \theta = \sin^{-1}\frac{\sqrt{3}}{2d}$, so

$$F_D(d) = \frac{2d^2}{3\sqrt{3}}\sin^{-1}\frac{\sqrt{3}}{2d} + \frac{\sqrt{4d^2-3}-1}{6}. \quad (10)$$

3) $\sqrt{3} \leq d \leq 2$: In this case, $\mathcal{S} = \frac{\pi d^2}{6} - 2\left(\frac{\theta}{2}d^2 - \frac{\sqrt{3}}{2}\sqrt{d^2-3}\right)$, where $\theta = \sin^{-1}\frac{\sqrt{d^2-3}}{d}$ according to Fig. 3(c). Thus

$$F_D(d) = \frac{d^2}{3\sqrt{3}}\left(\frac{\pi}{3} - 2\sin^{-1}\frac{\sqrt{d^2-3}}{d}\right) + \frac{2}{3}\sqrt{d^2-3}. \quad (11)$$

With (9)–(11), we have the corresponding probability density function of random distances, from one vertex of a hexagon to an arbitrary node inside the same hexagon,

$$f_{D_{\mathrm{V}}}(d) = \frac{2d}{3\sqrt{3}}\begin{cases} \frac{2\pi}{3} & 0 \leq d \leq 1 \\ 2\sin^{-1}\frac{\sqrt{3}}{2d} & 1 \leq d \leq \sqrt{3} \\ \frac{\pi}{3} - 2\sin^{-1}\frac{\sqrt{d^2-3}}{d} & \sqrt{3} \leq d \leq 2 \\ 0 & \text{otherwise} \end{cases}. \quad (12)$$

This is the scenario where the BS is located at the intersection point of three cells, using directional antenna for each of the adjacent cells. Also, during the cell-splitting phase in a cellular system, a new BS may be located at a vertex to improve the coverage and system capacity. When the BS is located at an arbitrary location, either outside or inside the hexagon, the distribution functions can also be derived following the same area-ratio approach. Here we omit their results due to the page limit.

Note that although the unit length is assumed in the above derivation, the distance distributions can be scaled by an arbitrary nonzero scalar. Let this arbitrary scalar be s, then

$$F_{sD}(d) = P(sD \leq d) = P(D \leq \frac{d}{s}) = F_D(\frac{d}{s}).$$

Therefore,

$$f_{sD}(d) = F'_D(\frac{d}{s}) = \frac{1}{s}f_D(\frac{d}{s}). \quad (13)$$

D. Verification of the Distance Distributions

Since the previous work has used the distance distributions of circles to approximate hexagons, we first compare these two sets of distribution functions. The inscribed or enclosing circle of a hexagon, either underestimates or overestimates the distance from the center, and thus the circle with the same area as the hexagon is also used [4], [5]. Suppose the side length of a hexagon is 1, the radius of the circle having the same size as the hexagon is $r = \sqrt{\frac{3\sqrt{3}}{2\pi}}$. For the inscribed and enclosing circle, r is $\frac{\sqrt{3}}{2}$ and 1, respectively.

If nodes are uniformly distributed over the cell area, the probability density function of random distances inside a circle of radius r, from the cell center, is

$$\widetilde{f}_{D_{\mathrm{I}}}(d) = 2d/r^2, \quad 0 \leq d \leq r, \quad (14)$$

and

$$\widetilde{f}_{D_{\mathrm{A}}}(d) = \frac{2d}{\pi r^2}\cos^{-1}\frac{d^2+R^2-r^2}{2Rd}, \quad R-r \leq d \leq R+r, \quad (15)$$

is for the distance from a point outside the circle, where $R = \sqrt{3}$ is the distance from the outside point to the circle's center for hexagons, according to [11]. The probability density function of the random distances from one fixed point on a circle to any point inside of it, is also given in [11],

$$\widetilde{f}_{D_{\mathrm{V}}}(d) = \frac{2d}{\pi r^2}\cos^{-1}\frac{d}{2r}, \quad 0 \leq d \leq 2r. \quad (16)$$

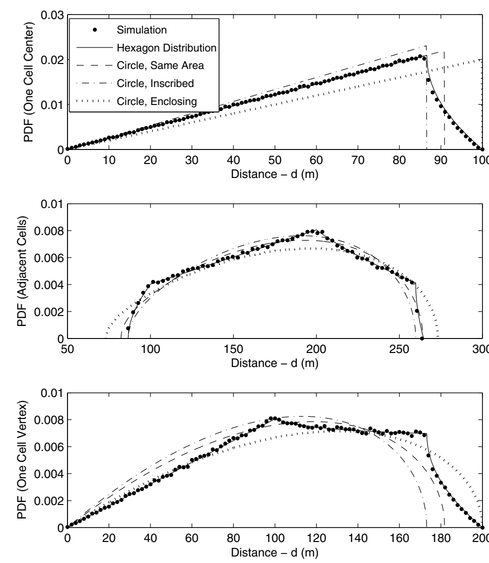


Fig. 4. Distance Distributions of Hexagons and Circles vs. Simulation.

Figure 4(a) compares (3) with (14), Fig. 4(b) compares (8) with (15), and Fig. 4(c) compares (12) with (16), assuming the cell size is 100 m. Figure 4(a)–(c) also compare the derived distributions with the results from Monte Carlo simulation. The distributions for circles are very different from those for hexagons, where the latter have the exact distance results as the simulation. This verifies the correctness of the derived hexagonal distance distribution functions.

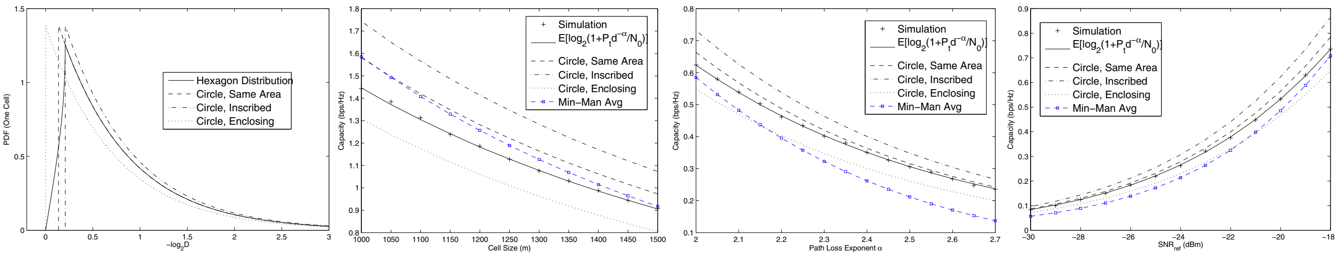
IV. Performance Study using Distance Distributions

Using the distance distribution functions derived in Section III, we can further build accurate analytical models for the per-user link capacity and the co-channel interference, for an AWGN channel. When compared with the exact hexagon distribution models, the traditional approximations, circular approximations or the min-max average, are found inaccurate in analyzing both the statistical distribution and the average of the system performance.

A. Per-User Link Capacity

For a node randomly deployed in a hexagonal cell communicating with the base station located at the center of the cell, the received signal-to-noise ratio (SNR) is $\mathrm{SNR} = \frac{P_t d^{-\alpha}}{N_0}$, where P_t is the transmission power, and α is the path loss exponent, and N_0 is the noise floor of an AWGN channel. Given the SNR, the capacity of the communication link is

$$C = B\log_2(1 + \mathrm{SNR}),$$



(a) Distribution of $\log_2 D$ (b) Capacity vs. Cell Size (c) Capacity vs. Path Loss Exponent (d) Capacity vs. Transmission Power

Fig. 5. Per-User Link Capacity.

where B is the channel bandwidth. C is determined by the following parameters in a high SNR environment,

$$C \approx B\left(\log_2 P_t - \alpha \log_2 D - \log_2 N_0\right),$$

which is a function of D when P_t, α and N_0 are constants. Different values of P_t and N_0 shift the range of C, and α expands or squeezes C. Let $Y = -\log_2 D$, with the distance distribution function $f_{D_\mathrm{I}}(d)$ in (3), we have

$$\begin{aligned} P(Y \le y) &= P(-\log_2 D \le y) \\ &= P(D \ge 2^{-y}) = 1 - F_{D_\mathrm{I}}(2^{-y}). \end{aligned}$$

Therefore,

$$f_Y(y) = \left[1 - F_{D_\mathrm{I}}(2^{-y})\right]' = \ln(2) 2^{-y} f_{D_\mathrm{I}}(2^{-y}). \quad (17)$$

On the other hand, the circular approximations replace $f_{D_\mathrm{I}}(2^{-y})$ in (17) by $\widetilde{f}_{D_\mathrm{I}}(2^{-y})$ in (14), with different values of r. Figure 5(a) shows the distribution of (17) with the hexagon model and circular approximations, where the circle with the same area size as the hexagon best approximates the hexagon distribution, because their PDFs are mostly overlapping. This gives us a hint that this particular circle may give good results for statistical moments, after being integrated. However, the exact distribution, instead of the expectation and other moments, determines the important system parameters especially when an individual user moves inside a cell. For instance, users at different locations have different capacities when communicating with the BS.

For the convenience of modeling and simulation, we give the analysis of the first moment, the expectation of link capacity. To compute this expectation, in the unit of per user per Hz, with the hexagon distribution $f_{D_\mathrm{I}}(d)$, we have

$$E\left[\log_2\left(1 + \frac{P_t x^{-\alpha}}{N_0}\right)\right] = \int \log_2\left(1 + \frac{P_t x^{-\alpha}}{N_0}\right) f_{D_\mathrm{I}}(x) dx, \quad (18)$$

by integrating over the cell area. Figure 5(b) compares the simulation results with the hexagon distribution model in (18), the circular approximations and the min-max average, assuming the path loss exponent $\alpha = 2$ and the BS transmission power $P_t = 50$ Watt. The figure shows that (18) is the most accurate model, whereas the other models either overestimate or underestimate the link capacity.

In Fig. 5(c), where α is used as the control parameter, the min-max average model is clearly much more sensitive to the path loss exponent, where the error grows with α. $\mathrm{SNR}_\mathrm{ref}$ in Fig. 5(d) is the received signal strength for nodes at the cell border, corresponding to P_t from 4 to 64 Watt. From the figures, as we anticipate, the link capacity decreases with the cell size and path loss exponent, but increases with the transmission power. The min-max average model deviates from the simulation results further when the cell size is smaller, or when α is higher. However, it is interesting to see from all three figures that, using a circle with the same area size as a hexagon for approximation, the circle provides the best approximation, as we can expect from Fig. 5(a). Our hexagon model is accurate for all the parameter settings.

B. Co-Channel Interference

Besides the signal from the intended transmitters in the same cell, interferences in the same frequency channel (e.g., in CDMA systems) also arrive at the BS from the undesired transmitters in other cells, and lead to the degradation of the system performance. In a CDMA system, while a mobile user connects to its serving BS, the interference it causes simultaneously at an adjacent BS is proportional to $P_t d^{-\alpha}$, where d follows the distribution $f_{D_\mathrm{A}}(d)$ in (8). Let $Z = D^{-\alpha}$, where D is the random variable for the distance d, we have

$$\begin{aligned} P(Z \le z) &= P(D^{-\alpha} \le z) \\ &= P(D \ge z^{-\frac{1}{\alpha}}) = 1 - F_{D_\mathrm{A}}(z^{-\frac{1}{\alpha}}). \end{aligned}$$

Therefore,

$$f_Z(z) = \left[1 - F_{D_\mathrm{A}}(z^{-\frac{1}{\alpha}})\right]' = \frac{1}{\alpha} z^{-\frac{1}{\alpha}-1} f_{D_\mathrm{A}}(z^{-\frac{1}{\alpha}}). \quad (19)$$

The circular approximations replace $f_{D_\mathrm{A}}(z^{-\frac{1}{\alpha}})$ in (19) by $\widetilde{f}_{D_\mathrm{A}}(z^{-\frac{1}{\alpha}})$ in (15). Figure 6(a) shows the distribution of (19) with the hexagon model and circular approximations, where the circle with the same area, as well as the inscribed circle, are both close to the distribution of $D^{-\alpha}$. But the former again best approximates the hexagon distribution, with respect to the overlapping region of the corresponding PDFs. The expectation of the received interference strength from a mobile user, at the BS of any six adjacent cells, using the distance distribution function $f_{D_\mathrm{A}}(d)$ in (8), is

$$P_t E[d^{-\alpha}] = P_t \int x^{-\alpha} f_{D_\mathrm{A}}(x) dx, \quad (20)$$

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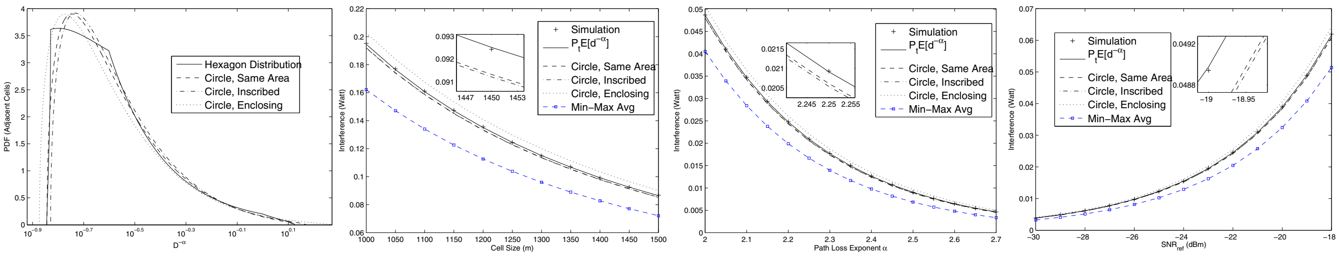

(a) Distribution of $D^{-\alpha}$ (b) Interference vs. Cell Size (c) Interference vs. Path Loss Exponent (d) Interference vs. Tx Power

Fig. 6. Co-Channel Interference.

by integrating over the adjacent cell area. Figure 6(b) compares the simulation results of the average interference with different models, with respect to the cell size. It is obvious that the model using the min-max average largely underestimates the average co-channel interference. While the two circular approximations in the zoom-in plot, i.e., using a circle with the same area as the hexagon and using an inscribed circle, both slightly underestimates the average interference. As shown in Fig. 6(a), their interference distribution functions are different from that of a hexagon. Although using a circle with the same area size still gives the best approximation, the slightly different distribution (even with the same average value) will affect important system parameters such as the link outage probability.

Figure 6(c) plots the average co-channel interference with different values of α, and Fig. 6(d) plots the results with different transmission power. The min-max average again underestimates the interference greatly in both figures. For instance, when $\alpha = 2.5$, the min-max model underestimates the interference by 50%. Note that even if the system is not CDMA-based, frequency reuse is unavoidable in such systems. In these situations, we can extend the above results by considering the co-channel interference from the cells which are non-adjacent to the tagged base station, and derive the co-channel interference distribution accordingly.

V. Conclusions

By deriving the closed-form distance distribution functions between cellular BSs and users inside the same, or adjacent hexagonal cells, this paper have developed a geometric probability model for analyzing the per-user link capacity and co-channel interference in cellular systems. The derivation technique, which is based on the area-ratio that is not limited by node coordinates, is simpler when compared with the existing methods in geometric probability, and the accuracy of the model has been validated through simulations. The approach can also be applied to deriving the distance distributions from a non-center point.

Our future work includes deriving distance distributions under general user distributions, such as Gaussian distributions, etc. The conditional probability of distances from an arbitrary point to both the serving BS and the interfering BS, can be derived and utilized to model the location-dependent interference factor. Furthermore, the distance distributions when both endpoints inside a cell, or between adjacent cells, become random, are more mathematically challenging [16], [17]. However, such distributions will be very useful, e.g., in analyzing hidden terminal problems and cooperative communications. We believe the probabilistic model proposed in this paper and its future extensions will provide guidelines for a more accurate network dimensioning and protocol design.

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