On Planar Tessellations and Interference Estimation in Wireless Ad-Hoc Networks

Yanyan Zhuang, T. Aaron Gulliver, and Yvonne Coady

***Abstract*—In wireless networks, the received signal power is related to the distance between the transmitter and receiver. Planar tessellations, such as square and hexagonal tessellations, have found a wide range of applications in the analysis of wireless networks. A geometric probabilistic model is considered for interference estimation in wireless ad-hoc networks, utilizing the distance distributions for hexagonal and square tessellations. The impact of employing square and hexagonal regions is analyzed. Simulation results demonstrate that both models are accurate.**

***Index Terms*—Geometric probability, distance distribution, planar tessellations, interference, hexagon, square.**

I. Introduction

SPATIAL reuse is a fundamental technique used to increase network capacity for the ever-increasing demands for wireless services. For this purpose, the network area is typically divided into congruent polygons, or cells. Such a partitioning is called a tessellation. Two planar tessellations with regular polygons are widely used with wireless networks: square [1] and hexagonal [2]–[4], as shown in Fig. 1. Many systems have been designed based on these tessellations. For example, sensor networks or mesh networks are typically partitioned using a square tessellation [1]. Cellular networks have also been designed based on hexagonal tessellations, as this closely approximates the circular radiation patterns of wireless signals. They can be used to partition a large area into adjacent, non-overlapping areas [8], and have the flexibility to be subdivided into smaller hexagons [5], or grouped together to form larger hexagons [6]. Further, hexagons have been shown to result in a substantial reduction in power consumption compared to using other tessellations [7]. Here, the primary application is wireless sensor networks (WSNs).

In wireless networks, node locations and inter-node distances are critical factors that affect system performance [8]. For instance, the k-th nearest neighbor distance is crucial for relay and routing protocols [9]. Stochastic mobility models are closely related to the trajectory between random points [10]. A closed-form distribution for random distances can also be applied to energy consumption in wireless sensor networks [1], path loss and link capacity in wireless communication networks [2], nearest and farthest neighbors [3], and transmission power given a fixed received power threshold [8]. The closed-form distribution for random distances is not only useful for deriving the interference distributions studied in this letter, but also other performance metrics in ad-hoc networks.

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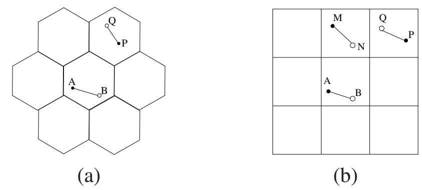

Fig. 1: Planar tessellations with (a) hexagons and (b) squares.

The distance distributions in and between regular hexagons were considered in [8]. Previous results on hexagonal tessellations considered a fixed reference point in the network [2]. However, those approaches only apply to communications between random nodes and this fixed point. Although this is typical in a cellular system, it is not the case for more complex situations such as in WSNs. This letter presents the derivation of closed-from distance distributions for hexagonal tessellations when both endpoints of a link are randomly located.

The interference between nodes is a key issue impacting the performance in a WSN, such as link capacity and throughput. Therefore, the distribution results obtained here for hexagons and the known distributions for squares are used to develop an analytic interference model for these networks with hexagonal and square tessellations. The accuracy of the model is verified via simulation.

II. Background and Related Work

A. The Geometric Distribution of Random Distances

The study of the distribution of node distances dates back to the 1940's [21], [22]. The problem of deriving the *expected* distance between random points was listed as problem number 75-12 in SIAM Review [23]. On the other hand, obtaining the *distribution* of random distances, which leads to all statistical moments of the distances, is a challenging problem. This distribution is very useful, for example, in solving communication networks problems. Random distances when one of the endpoints is fixed have also been examined [14], and the *chord length* distribution has been studied when both endpoints are on the region boundary [15]. However, the problem becomes particularly difficult when both endpoints are random.

In [24], the Crofton technique and its extensions were used to obtain the geometric distribution for circles and squares. Distance distributions were derived in [11], [16], [25] for a few simple geometric cases. All results to date have either considered random distances from a fixed reference point, or provide distributions for very specific network topologies [11]. In particular, the independence of node coordinates is often required [1], [21], [22]. However, this assumption does not hold in a hexagonal topology.

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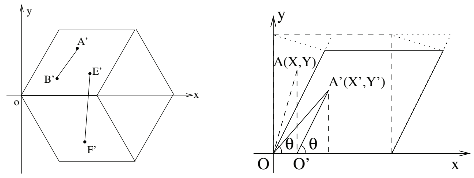

(a) (b)

Fig. 2: The relationship between (a) a rhombus and a hexagon, and (b) a square and a rhombus.

B. Distance Distributions and Wireless Networks

In a spatially random network, wireless devices are typically distributed over an area according to a given distribution. The distance between these devices (nodes) plays an important role in determining system performance. In [17], the angle of arrival (AoA) of interfering signals at the *center* of a cell from adjacent hexagonal cells was studied for the uplink of a cellular system. A first-order linear approximation for the interference within a single hexagon was also given. An adjacent cell interference model between two base stations was presented in [18], but a closed-form expression was not obtained. The per-user capacity and co-channel interference in a cellular system was analyzed in [2] assuming a base station as a fixed endpoint in the network.

Haenggi and Ganti [19] also considered interference in tessellated networks, but with both transmitters and receivers located at predefined locations. The stochastic geometry approach in [20] considered infinite networks where node coordinates follow a Poisson point process. In practice, however, all networks have a finite size, and thus the results given in [20] can serve only as theoretical upper bounds. Conversely, the model presented here can be used to obtain exact results for finite networks. Furthermore, all previous results are either limited to a single hexagon, or adjacent hexagons with a fixed endpoint. No analytic model has yet been presented for the interference in a network where both endpoints are randomly located in the hexagon(s).

III. Random Distances for Regular Hexagons

By dividing a regular hexagon into three congruent rhombuses as in Fig. 2(a), the distribution functions for the random distances for hexagons can be derived explicitly.

A. Preliminaries

The Euclidean distance between random points (X_1, Y_1) and (X_2, Y_2) is $D = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}$. Let the random variable Z denote the squared Euclidean distance D^2, $X = X_1 - X_2$ and $Y = Y_1 - Y_2$, so that $Z(X, Y) = X^2 + Y^2$. For given X and Y, the distribution of Z is

$$P(Z \le z) = \iint P\left(Z(X,Y) \le z | X = x, Y = y\right) f_{X,Y}(x,y)dxdy, \tag{1}$$

where $f_{X,Y}(x, y)$ is the joint probability density function. Henceforth, we use X to denote a random variable, x for a sample value of X, and $\mathcal{X}$ to denote the set of all possible values of X. It is assumed that the probability distribution of the random points is uniform, where $U[a, b]$ is used to denote a uniform distribution over the interval $[a, b]$.

B. Random Distances for a Rhombus

Consider the square in Fig. 2(b). The point $\mathrm{A}(X, Y)$ forms a right triangle OAO' with the $\mathcal{X}$-axis. Skewing the square by $\frac{\pi}{2} - \theta$ (assuming $0 \le \theta \le \frac{\pi}{2}$), $\mathrm{A}(X, Y)$ in the square becomes $\mathrm{A}'(X', Y')$ in a rhombus, which forms an obtuse triangle $\mathrm{OA'O'}$ with the $\mathcal{X}$-axis. The relation between A and A$'$ is

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} 1 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}, \quad \text{or} \quad \begin{cases} X' = X + Y\cos\theta \\ Y' = Y\sin\theta \end{cases}, \tag{2}$$

where (X_1', Y_1') and (X_2', Y_2') ((X_1, Y_1) and (X_2, Y_2)) are random variables denoting the coordinates after (and before) the transformation in (2). For two points in a rhombus, the squared Euclidean distance is

$$\begin{aligned} Z &= D^2 = (X_1' - X_2')^2 + (Y_1' - Y_2')^2 \qquad (3) \\ &= (X_1 - X_2)^2 + 2\cos\theta(X_1 - X_2)(Y_1 - Y_2) + (Y_1 - Y_2)^2, \end{aligned}$$

where (X_1, Y_1) and (X_2, Y_2) are the coordinates in the original square. Because $X = X_1 - X_2$ and $Y = Y_1 - Y_2$, then $Z = X^2 + 2\cos\theta XY + Y^2$. This is the implicit equation of a non-degenerate real ellipse when $\theta \ne \frac{\pi}{2}$ [12].

Define a *unit rhombus* as the rhombus with an acute angle $\theta = \frac{\pi}{3}$ and a side length of 1. The coordinates of a point in this rhombus are $X_1' = X_1 + \frac{Y_1}{2}$ and $Y_1' = \frac{\sqrt{3}}{2}Y_1$, where X_1 and Y_1 are the coordinates in the original unit square. Assuming X_1 and Y_1 are uniformly distributed over $U[0, 1]$, then (3) becomes $Z = X^2 + XY + Y^2$, with X and Y both following a triangular distribution. The distribution of Z is

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X^2 + XY + Y^2 \le z) \\ &= \iint P(X^2 + XY + Y^2 \le z | X = x, Y = y) \\ &\qquad f_{X,Y}(x,y)dxdy. \end{aligned} \tag{4}$$

For a square, X and Y are independent so $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, where $f_X(x)$ and $f_Y(y)$ are the (marginal) probability density functions of X and Y. Denote the shape $X^2 + XY + Y^2 \le z$ as $\Omega(z)$. Since $Z = X^2 + XY + Y^2$ is a function of an ellipse, $\Omega(z)$ is a series of elliptical contours, with the size dependent on the value of z. Therefore, (4) can be written as $F_Z(z) = \iint_{\Omega(z)} f_X(x)f_Y(y)dxdy$, and its probability density function is $f_Z(z) = F_Z'(z)$. With $D = \sqrt{Z}$, the distance distribution is

$$f_D(d) = F_Z'(d^2) = 2df_Z(d^2). \tag{5}$$

The derivation of this result can be found in [8].

C. Random Distances for Regular Hexagons

1) Distance Distribution within a Hexagon: Define a *unit hexagon* as a regular hexagon with a side length of 1. For a line inside a unit hexagon, as in Fig. 2(a), there are two cases: i) both endpoints fall in the same unit rhombus, e.g. $A'B'$, with probability $\frac{1}{3}$; and ii) the endpoints fall in adjacent rhombuses sharing a side, e.g. $E'F'$, with probability $\frac{2}{3}$. Denoting the distribution for $|A'B'|$ as $f_{D_{\mathrm{R}_1}}(d)$, and for $|E'F'|$ as $f_{D_{\mathrm{R}_2}}(d)$, the probability density function of the random Euclidean distances between two endpoints in a hexagon is

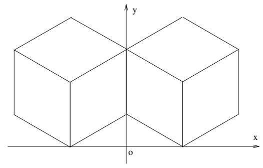

Fig. 3: Two adjacent hexagons and rhombuses.

$f_{D_{\mathrm{H_I}}}(d) = \frac{1}{3} f_{D_{\mathrm{R_1}}}(d) + \frac{2}{3} f_{D_{\mathrm{R_2}}}(d)$. $f_{D_{\mathrm{R_1}}}(d)$ has been derived in Section III-B.

Using the same transformation as in (2), and denoting the coordinates as $\mathrm{E}'(X_1', Y_1')$ and $\mathrm{F}'(X_2', Y_2')$, gives $X_1' = X_1 + \frac{Y_1}{2}$, $Y_1' = \frac{\sqrt{3}}{2} Y_1$, $X_2' = X_2 + \frac{Y_2}{2}$, and $Y_2' = -\frac{\sqrt{3}}{2} Y_2$. We assume X_1, Y_1, X_2 and Y_2 are uniformly distributed over $U[0,1]$. The squared Euclidean distance Z is then given by

$$\begin{aligned} Z &= (X_1' - X_2')^2 + (Y_1' - Y_2')^2 \qquad (6) \\ &= (X_1 - X_2)^2 + (X_1 - X_2)(Y_1 - Y_2) + Y_1^2 + Y_1 Y_2 + Y_2^2. \end{aligned}$$

Let $X = X_1 - X_2$, then

$$\begin{aligned} Z &= X^2 + X(Y_1 - Y_2) + Y_1^2 + Y_1 Y_2 + Y_2^2 \\ &= Y_1^2 + Y_2^2 + X^2 + Y_1 Y_2 + Y_1 X - Y_2 X. \qquad (7) \end{aligned}$$

The RHS of (7) satisfies the general form of a non-degenerate quadratic surface [13]. Therefore, the shape $\Omega(z)$ of (7) for different values of z are concentric cylinders. Equation (4) then becomes a triple integral

$$F_Z(z) = \iiint_{\Omega(z)} f_X(x) f_{Y_1}(y_1) f_{Y_2}(y_2) dx dy_1 dy_2. \qquad (8)$$

Therefore, $f_{D_{\mathrm{R_2}}}(d)$ can be derived from (8) and (5), the details of which can be found in [8].

2) Distance Distribution between Adjacent Hexagons: The distance distribution when the random endpoints are located in adjacent hexagons, denoted as $f_{D_{\mathrm{H_A}}}(d)$, is more complex. As shown in Fig. 3, when one endpoint is fixed inside a hexagon, the other endpoint can fall inside any of the three rhombuses in the adjacent hexagon. Therefore, there are 3×3 different cases. Some of these cases are identical because of the invariance properties of the distance measures [11][1]. The derivation of $f_{D_{\mathrm{H_A}}}(d)$ can be found in [8].

Although unit hexagons have been assumed, the distance distribution can be scaled by a nonzero scalar to obtain the distribution for a hexagon of arbitrary size. Let this scalar be s, then $F_{sD}(d) = P(sD \le d) = P(D \le d/s) = F_D(d/s)$, and

$$f_{sD}(d) = F_D'(d/s) = f_D(d/s)/s. \qquad (9)$$

IV. Interference Estimation in Planar Ad-Hoc Wireless Networks

In this section, the interference in both hexagon and square tessellated networks is analyzed. We assume the 7 hexagons in Fig. 1(a) have the same total area as the 9 squares in Fig. 1(b). If the side length of a hexagon is s, then the side length of a square s' is $\sqrt{\frac{7\sqrt{3}}{6}} s$. Thus a square in Fig. 1(b) is smaller than a hexagon in Fig. 1(a). In both figures, dots represent transmitters and circles represent receivers. In an ad-hoc network, the transmitter-receiver pairs are randomly located. When A communicates with B, the interference level at B is affected by the simultaneous transmissions of: i) nodes within the same cell, and ii) nodes in adjacent cells.

[1]The distance between two points is not changed when relocating the origin of the coordinate axes. Even under rotation and reflection of the axes, the distance does not change.

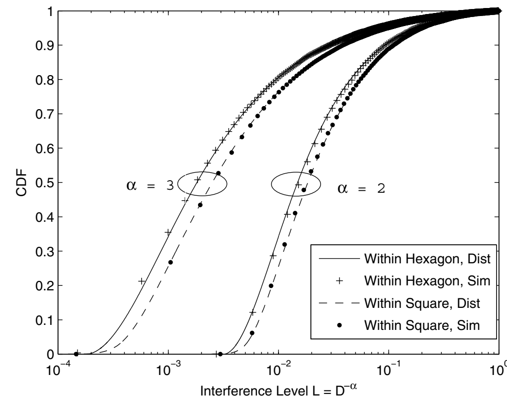

Fig. 4: Interference distribution within a cell.

A. Interference within a Cell

We first consider a single cell, e.g. the middle cell in Fig. 1(a) and (b), with a pair of randomly located nodes A and B. While A and B communicate, the interference at receiver B caused by another transmitter (interferer) in the *same* cell can be expressed as $I = P_t d^{-\alpha}$, where P_t is the transmission power, d is the distance between the interferer and receiver, and α is the path loss exponent. With a hexagonal tessellation, d follows the distribution $f_{D_{\mathrm{H_I}}}(d)$ in Section III-C1. Without loss of generality, we assume P_t is one power unit, and let $L = D^{-\alpha}$, where D is the random variable for distance d. The distribution of the interference inside a cell is then

$$\begin{aligned} F_{L_I}(l) &= P(L \le l) = P(D^{-\alpha} \le l) \\ &= P\left(D \ge l^{-\frac{1}{\alpha}}\right) = 1 - F_{D_{\mathrm{H_I}}}\left(l^{-\frac{1}{\alpha}}\right), \qquad (10) \end{aligned}$$

and thus

$$f_{L_I}(l) = \left[1 - F_{D_{\mathrm{H_I}}}\left(l^{-\frac{1}{\alpha}}\right)\right]' = \frac{1}{\alpha} l^{-\frac{1}{\alpha}-1} f_{D_{\mathrm{H_I}}}\left(l^{-\frac{1}{\alpha}}\right). \qquad (11)$$

If A and B are located in a square as in Fig. 1(b), then $f_{L_I}(l)$ can be obtained by replacing $f_{D_{\mathrm{H_I}}}$ in (11) with the distance distribution within a square (derived by Ghosh in [21]).

Fig. 4 shows the CDF of the interference distribution, i.e., $L = D^{-\alpha}$ assuming $P_t = 1$, and the corresponding simulation results. The analytic and simulation results were both obtained using Matlab. A side length of $s = 100\ m$ was used for a hexagon and the corresponding value of s' for a square. Path loss exponents $\alpha = 2$ and 3 were considered. These results verify the analytic model.

B. Interference between Adjacent Cells

Replacing $f_{D_{\mathrm{H_I}}}$ in (11) by $f_{D_{\mathrm{H_A}}}(d)$ from Section III-C2, the interference between a pair of nodes in *adjacent* hexagonal cells (e.g. from interferer P to receiver B in Fig. 1(a)), is

$$f_{L_A}(l) = \frac{1}{\alpha} l^{-\frac{1}{\alpha}-1} f_{D_{\mathrm{H_A}}}\left(l^{-\frac{1}{\alpha}}\right). \qquad (12)$$

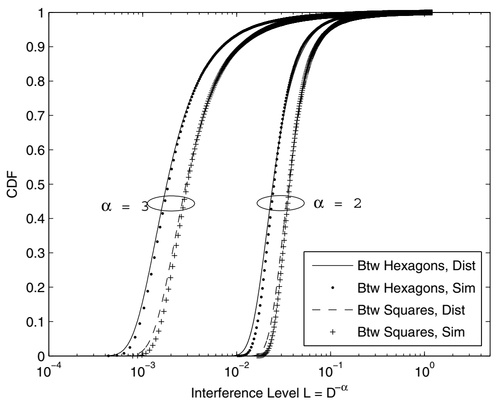

Fig. 5: Interference distribution between adjacent cells.

Because of symmetry, from receiver B any interferer in an adjacent hexagonal cell is equivalent to P. Therefore, to obtain the interference from all six adjacent cells in a hexagonal tessellation, a 6-fold convolution of (12) is required

$$f_{L_A}^{\text{hex}}(l) = f_{\sum_{i=1}^{6} L_{Ai}}(l) = f_{L_A}^{6^*}(l), \tag{13}$$

where each $f_{L_{Ai}}(l)$ has the same distribution as (12). Equation (13) is the probabilistic distribution of the accumulated interference from the transmitters in all six adjacent cells to a receiver in the center cell.

When interferers are located in adjacent squares, two separate cases need to be considered, i.e., PQ and MN in Fig. 1(b). The total interference at receiver B is obtained by first convolving the interference caused by M and P

$$f'_{L_A}(l) = \left[\frac{1}{\alpha} l^{-\frac{1}{\alpha}-1} f_{D_{\mathrm{S_P}}}\left(l^{-\frac{1}{\alpha}}\right)\right] * \left[\frac{1}{\alpha} l^{-\frac{1}{\alpha}-1} f_{D_{\mathrm{S_D}}}\left(l^{-\frac{1}{\alpha}}\right)\right], \tag{14}$$

where $f_{D_{\mathrm{S_P}}}$ and $f_{D_{\mathrm{S_D}}}$ are the distance distributions of two endpoints located in adjacent squares that are sharing a side, and adjacent squares sharing a common diagonal, respectively (these distributions were derived by Ghosh [22]). A 4-fold convolution of (14) gives the result for a square tessellation

$$f_{L_A}^{\text{sqr}}(l) = f'_{\sum_{i=1}^{4} L_{Ai}}(l) = f'^{4^*}_{L_A}(l). \tag{15}$$

Fig. 5 shows the interference distribution of $L = D^{-\alpha}$ from adjacent cells for square and hexagonal tessellations. The side length of each hexagon is again $s = 100\ m$, and the path loss exponents was $\alpha = 2$ and 3. From this figure, the accumulated interference from the adjacent square cells is higher than from adjacent hexagonal cells, and the difference is greater than the interference within the same cell. Thus, it can be concluded that in terms of wireless interference, hexagonal tessellations have better geometric properties than square tessellations. Since the interference between wireless devices is a major factor affecting network performance, hexagonal tessellations will provide higher capacity and throughput.

V. Conclusion

In this letter, an analytic interference model was presented for hexagonal and square tessellated networks. This model can be used to evaluate network capacity and throughput. Therefore it is a valuable tool for network protocol and system design.

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